An analysis of North Carolina's legislative districtings: Expert Report

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1 Qualifications

I am an associate professor in the department of Mathematical Sciences at Carnegie Mellon University, where I have been a member of the faculty since 2013. I received my Ph.D. in Mathematics from Rutgers University in 2010 under the supervision of József Beck, and I am an expert on stochastic processes and discrete probability. My research has been funded by the National Science Foundation and the Sloan Foundation. A current CV with a list of publications is attached as Exhibit A. A list of my publications with links to online manuscripts is also available at my website at `http://math.cmu.edu/~wes`.

I am an expert on the use of Markov Chains for the rigorous analysis of gerrymandering, and have published papers[1] developing techniques for this application in *Proceedings of the National Academy of Sciences* and *Statistics and Public Policy*, hereafter referred to by [CFP] and [CFMP], respectively.

I testified as an expert witness in the *League of Women Voters of Pennsylvania v. Commonwealth of Pennsylvania* case in which the 2011 Congressional districting was found to be an unconstitutional partisan gerrymander, and as well as the *Common Cause v. Lewis* case in North Carolina. I previously served as a member of the bipartisan Pennsylvania Redistricting Reform Commission under appointment by the governor. I am being compensated at a rate of $325 per hour for my work on the current case.

2 Executive Summary

I was asked to analyze whether the proposed Congressional, state House, and state Senate districtings of North Carolina were drawn in a way which made extreme use of partisan considerations.

To conduct my analysis, I take the enacted plan as a starting point and make a sequence of many small random changes to the district boundaries. This methodology is intended to detect whether the district lines were carefully drawn to optimize partisan considerations; in particular, if the plans in question were not intentionally drawn to maximize partisan advantage, then making random changes should not significantly decrease the plan's partisan bias.

Specifically, my method begins with the enacted plan and uses a Markov Chain—a sequence of random changes—to generate trillions of comparison districtings against which I compare the enacted plans. These comparison districtings are generated by making a sequence of small random changes to the enacted plans themselves, and preserve districting criteria such as population deviation, compactness, and splitting of counties, municipalities, and precincts, among other criteria (a complete list is given in Section 4.3.1).

The analysis I conduct of the enacted plan using this data has two levels. The first level of my analysis consists simply of comparing the partisan properties of the enacted plans to the large sets of comparison maps produced by my Markov Chain, and I report how unusual the enacted plans are with respect to their partisan properties, against this comparison set. **Quantitatively, for the enacted Congressional, House, and Senate plans, I find that they have a greater partisan bias than 99.99999%, 99.99999%, and 99.97% of the trillions of districtings produced by my algorithm, respectively.**

The next level of my analysis uses the mathematical theorems I have developed with my co-authors in [CFP] and [CFMP] to translate the results of the above comparison into a statement about how the enacted plans compare against *all* other districtings of North Carolina satisfying the districting criteria I consider in this report. In other words, the theorem that I use in the second level analysis allows me to compare the enacted plan against not only the trillions of plans that my simulations produce through making small random changes, but also against all other possible districtings of North Carolina satisfying the districting criteria I consider.

Consider the following: when I make a sequence of small random changes to an enacted plan as described above, this can be viewed as a test of whether the partisan bias in the current districting is fragile, in the sense that it evaporates when the boundary lines of the district are perturbed. As discussed in Section B, our

[1]

- M. Chikina, A. Frieze, W. Pegden. Assessing significance in a Markov Chain without mixing, in *Proceedings of the National Academy of Sciences* **114** (2017) 2860–2864
- M. Chikina, A. Frieze, J. Mattingly, W. Pegden. Separating effect from significance in Markov chain tests, in *Statistics and Public Policy* **7** (2020) 101–114.

theorems in [CFP] and [CFMP] establish that it is mathematically impossible for the political geography of a state to cause such a result. That is: while political geography might conceivably interact with districting criteria to create a situation where typical districtings of a state are biased in favor of one party, it is mathematically impossible for the political geography of a state to interact with districting criteria to create a situation where typical districtings of a state appear to be *optimized for partisan bias*, in the sense that their bias is fragile and evaporates when small random changes are made. This allows us to rigorously demonstrate that a districting is optimized for partisanship, and is an outlier among *all* districtings of a state satisfying the criteria I consider, with respect to this property.

Quantitatively, my second-level analysis establishes that the enacted plans here are more optimized for partisanship than 99.9999% of all possible Congressional districtings satisfying the districting criteria I account for in my analysis, more than 99.9999% of all possible House districtings satisfying those criteria, and more than 99.9% of all Senate districtings satisfying those criteria. Thus the chance of drawing districtings that are as optimized with respect to their partisan properties as the current House and Senate districtings of North Carolina *without* using partisan considerations is exceedingly small.

In particular, I find that **North Carolina's Congressional, House and Senate districtings were drawn in a way which made extreme use of partisan considerations, a finding which is mathematically impossible to be caused by the interaction of political geography and the districting criteria I consider.**

3 Topic of Expert Report

The question motivating my analysis in this case is: "*How significant a role did partisanship play in the drawing of the enacted Congressional, House and Senate districts of North Carolina?*"

My analysis approaches this question in a rigorous and quantifiable way. In short, I identify how much of an outlier the present districting lines are, with respect to how carefully they are drawn to line up with partisan goals. *A priori*, it is possible that political geography might conceivably interact with districting criteria to bias typical districtings for one party or another. But my analysis provides a rigorous quantifiable answer to the question of the extent to which partisanship was used in the districting process, whose validity *does not depend on the political geography of North Carolina*.

Apart from whole-state analyses of the enacted Congressional, House and Senate plans of North Carolina, I was also asked to conduct separate analyses of the following specific House and Senate clusters:
House:

- Mecklenburg
- Wake
- Forsyth-Stokes
- Guilford
- Buncombe
- Pitt
- Duplin-Wayne
- Alamance
- Durham-Person
- Cumberland
- Cabarrus-Davie-Rowan-Yadkin
- Brunswick-New Hanover

Senate:

- Iredell-Mecklenburg
- Granville-Wake
- Forsyth-Stokes
- Cumberland-Moore
- Guilford-Rockingham

4 Quantifying intentional and excessive use of partisanship

My approach begins with a simple idea: I make small random changes to the boundaries of enacted plans (while maintaining districting criteria) and study the effect this has on the partisan bias of the map. More specifically:

- I begin from the enacted plan I am evaluating, and then repeatedly:
  1. Randomly select a geographical unit (e.g., a voting precinct) on the boundary of two districts, and check: if I change which district this geographic unit belongs to, will the resulting districting still satisfy the districting criteria laid out in Section 4.3.1? If so, I make the change.
  2. Using historical voting data as a proxy for partisan voting patterns, evaluate the partisanship of the districting resulting from the previous step.
- These two steps are repeated many times, resulting in a sequence of districtings, each produced by a small random change to the districting preceding it, with the enacted map I am evaluating as the starting point for the sequence.

This procedure is implemented as a computer algorithm which carries out trillions of the above steps for a districting map.

4.1 First level analysis

The first level of my analysis simply uses the above procedure to generate a large set of comparison districtings against which one can compare the enacted plan. For example, for the Congressional districting, I conducted 32 runs of the above procedure. A "run" in this context consists of a single consecutive sequence of small random changes to the enacted plan, producing a set of comparison districtings. For example, for the Congressional districting, each run consisted of carrying out Steps 1 and 2 in the procedure above $2^{40} \approx 1$ trillion times. As discussed in later sections, these comparison maps adhere to districting criteria in ways that constrain them to be similar in several respects to the enacted map being evaluated. For example, the comparison districtings will preserve the same counties and municipalities preserved by the enacted plan.

In total for this districting, I conducted 32 such runs. I then show the results of these runs in a table, like this:

Congressional districting

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999947%	9	99.9999909%	17	99.9999955%	25	99.999995%
2	99.999968%	10	99.99999966%	18	99.9999973%	26	99.9999961%
3	99.9999988%	11	99.9999943%	19	99.99999972%	27	99.99999977%
4	99.99999931%	12	99.999988%	20	99.9999999981%	28	99.99999979%
5	99.99999999927%	13	99.999988%	21	99.9999999962%	29	99.9999981%
6	99.9999959%	14	99.9999987%	22	99.99999919%	30	99.9999941%
7	99.99999984%	15	99.999996%	23	99.9999908%	31	99.99999901%
8	99.9999999947%	16	99.999985%	24	99.999981%	32	99.9999969%

For example, we see here that in the first run, 99.9999947% of the comparison districtings exhibited less Republican bias than the enacted Congressional districting. Moreover, in *every* run, more than 99.999968% of the comparison districtings exhibited less Republican bias than the enacted plan.

The first level of my analysis simply reports this comparison of the enacted map to the comparison districtings produced in these runs. Even without applying the mathematical theorems we have developed in [CFP] and [CFMP], this gives strong, intuitively clear evidence that the district lines were intentionally drawn to optimize partisan advantage in the enacted plan: if the districting had not been drawn to carefully optimize its partisan bias, we would expect naturally that making small random changes to the districting would not have such a dramatic and consistent partisan effect.

4.2 Second level analysis

In the first level of my analysis, I compare enacted plans to comparison districtings produced by my algorithm (which makes random changes to the existing map while preserving districting criteria).

The next level of my analysis goes further than this, and enables a rigorous comparison to *all* alternative districtings of North Carolina satisfying the districting criteria I consider here. It does this by comparing how "optimized for partisanship" an evaluated plan is to how "optimized for partisanship" alternative plans are.

4.2.1 Defining "optimized for partisanship"

Roughly speaking, when I say that a districting is *optimized for partisanship*, I mean that its partisan characteristics are highly sensitive to small random changes to the boundary lines.

Formally, when I say that a districting is *optimized for partisanship* in this report, I mean that there is a high probability that when I make small random changes to the districting, its partisanship will be an extreme outlier among the comparison maps produced by the small random changes.

The yardstick I use to measure this property of a given map is the *ε-fragility* of a map. Given a small threshold ε—for example, 00.000031%, for the analysis of the Congressional districting given above—I can ask: what is the probability that when I make a sequence of small random changes to the map, the map will be in the most extreme ε fraction of maps encountered in the sequence of random changes? The probability of this occurrence is the ε-fragility of the map, and it is this probability that I use to quantify how optimized for partisanship a map appears to be.

In other words, **one districting is considered more optimized for partisanship than another if it is more likely to have its partisan bias consistently reduced when making a random sequence of small changes to its boundary lines.**

4.2.2 Comparing an enacted plan to the set of *all* alternatives

My analysis enables a rigorous comparison of an enacted plan to **all possible districting plans of the state** satisfying the districting criteria I consider, with respect to how optimized for partisanship the districtings are. I can report the maximum fraction of all such possible redistricting plans which could appear as optimized for partisanship as the enacted plan, in the sense of the test described above. For example, I report that the enacted Congressional districting of North Carolina is among the most optimized-for-partisanship 00.000031% of **all possible House districtings of North Carolina** satisfying the districting criteria I consider here, as measured by it's ε-fragility.

My method produces a rigorous p-value (statistical significance level) which precisely captures the confidence one can have in the findings of my "second level" analyses. In particular, for my statewide analyses, my second-level claims are all valid at a statistical significance of $p = .002$. This means that the probability that I would report an incorrect number (for example, claiming that a districting is among the most optimized-for-partisanship 00.01% of all districtings, when in fact it is merely among the most 00.015% optimized-for-partisanship) is at most 00.2%. To put this in context, clinical trials seeking regulatory approval for new medications frequently target a significance level of $p = .05$ (5%), a looser standard of statistical significance than I hold myself to in this report.

4.2.3 Some intuition for why this is possible

It may seem remarkable that I can make a rigorous quantifiable comparison to *all* possible districtings, without actually generating all such districtings; this is the role of our theorems from [CFP] and [CFMP], which have simple proofs which have been verified by the mathematical community.

To give some nontechnical intuition for why this kind of analysis is possible, these results roughly work by showing that in a very general sense, it is not possible for an appreciable fraction of districtings of a state to appear optimized for partisanship in the sense defined in Section 4.2.1. In other words, it is *mathematically impossible* for any state, with any political geography of voting preferences and any choice of districting criteria, to have the property that a significant fraction of the possible districtings of the state satisfying the chosen districting criteria appear optimized for partisanship (as measured by their ε-fragility).

4.3 Implementation details

Here I specify the particulars of the random changes my algorithm makes to a map, my implementation of districting criteria, and my method of comparing the partisanship of a districting to that of districtings encountered on the sequence of random changes.

4.3.1 Districting criteria

All comparison maps produced by my algorithm are required to satisfy the following districting criteria:

(a) **Contiguity**: I require comparison districtings to contain only contiguous districts.

(b) **Compact districts**: I require comparison districtings to be at least as compact as the enacted plan being evaluated, up to an error of 5%. Districting compactness is quantified by taking the average, over each district, of the ratio of the perimeter squared to the area (Polsby-Popper reciprocal).

(c) **County clusters**: For the House and Senate plans, I require comparison maps to respect the same county clustering as used by the enacted House and Senate plans.

(d) **Country traversals**: I require comparison districts to not contain more county traversals than the enacted plan. Additionally, I constrain the total length of all district boundary which is not also county boundary to be at most that of the enacted map, up to an error of 5%.

(e) **Municipality preservation:** There are at most as many municipal splits as in the enacted plan.

(f) **VTD preservation:** The total number of VTD splits in comparison districtings must not exceed the total number of VTD splits in the enacted plan.

(g) **Incumbency protection:** Any incumbent who, in the enacted plan, is not paired with any other incumbent must remain unpaired in the comparison districtings.

(h) **Population deviation**: For House and Senate districtings, I require comparison districtings to have district populations within 5% of the ideal district population. For the Congressional districting, I use a 2% threshold in my main analysis. I discuss robustness of my Congressional analysis to differences in population criteria in Section 5.0.2. Population is measured by the 2020 decennial Census.

4.3.2 A conservative application of the criteria

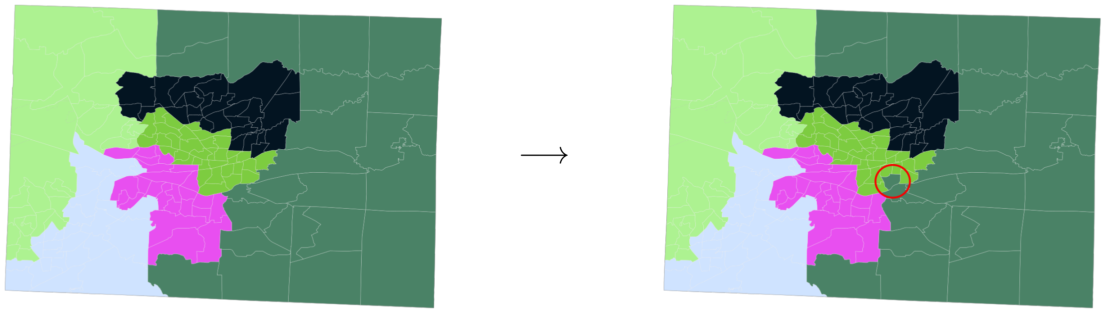
It is important to note that my analysis is designed to avoid second-guessing the mapmakers' choices in how they implemented the districting criteria. In particular, while it is reasonable to ask whether the mapmakers could have drawn districtings which adhered better to nonpartisan criteria (more compact, preserving more municipalities, *etc*), my approach is different, and much more conservative.

In particular, my analysis asks the question: even if we accept that the mapmakers have made appropriate choices with respect to nonpartisan criteria such as compactness, population deviation, municipality preservation, incumbency protection, and so on, does their plan nevertheless stand out with respect to its *partisan* qualities?

Note that, for example, I choose my compactness threshold within 5% of value of the enacted map. And with respect to incumbents, I do not try to protect as many incumbents as are protected in the enacted map, but exactly the same incumbents as protected by the mapmakers. With respect to municipality preservation, I am not trying to answer the question: "if the mapmakers had tried to preserve more municipalities, would this have resulted in a more favorable districting for Democrats?" Instead, I am asking, among all alternative districtings of North Carolina with the same *nonpartisan* characteristics as the enacted map—their compactness, how many municipalities they preserve, etc.—whether the enacted plan is an extreme outlier with respect to the extent to which it is optimized for partisanship.

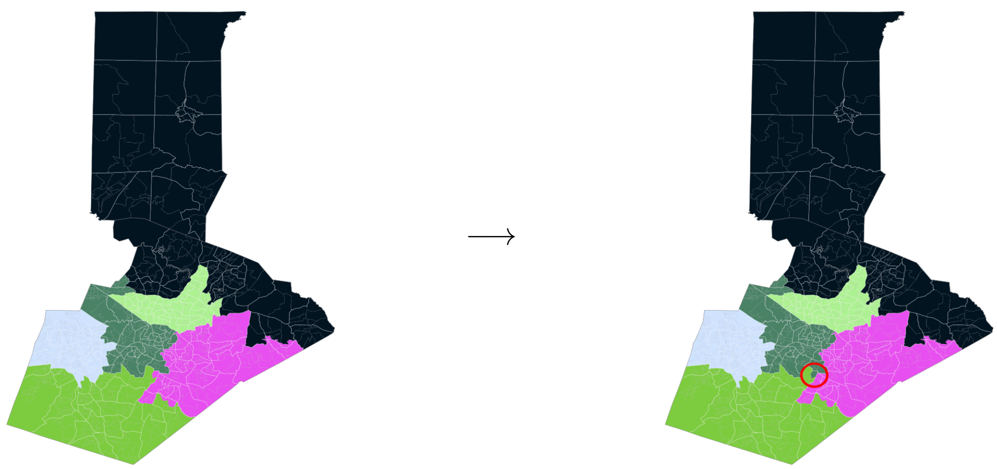
5 Random Changes

As described earlier, my method involves making small random changes to a map. For example, depicted here is a small random change made to the enacted House districting within the Guilford county cluster:



The geographical units used for these small random changes in this district are voting tabulation districts—VTDs. In particular, at each step of the sequence of random changes for the house districting within Guilford county, I move a randomly VTD that is at the boundary of two districts from one of those districts to the other (unless it would violate the constraints laid out in Section 4.3.1.

For House and Senate clusters that split VTDs, my analysis operates below the VTD level. In particular, my procedure in these case manipulates sub-VTD units (referred to hereafter as geounits). These are compact combinations of Census Blocks which respect VTD and district lines and contain on average approximately 1000 people. In particular, there are an average of around 4 geounits per VTD. In the following example from the Granville-Wake senate districting, we see an example of a random change at the geounit level:



The thick white lines here indicate current VTD boundaries. A geounit within an already broken VTD has changed district membership. When analyzing any districting at the below-VTD level, my algorithm constrains comparison maps to split at most as many VTDs as the enacted map.

For my whole-state analyses, my algorithm operates at the VTD level. This means that the algorithm is prohibited from splitting any VTD's not split in the enacted map. In Section C, I include runs where the Congressional districting is analyzed at the geounit level.

In each run, my chain generates comparison maps from a given enacted plan by making billions or trillions of these small changes to the enacted plan, while preserving districting criteria in specific ways chosen by the mapmakers, as discussed in Section 4.3.2.

These random changes can be either be made one-at-a-time or with several steps made simultaneously; the latter allows comparison maps to be generated when any single move would lead to a violation of the constraints laid out in Section 4.3.1 (e.g., because population would become too imbalanced), but combinations of moves can be found which would preserve all these criteria. My mathematical analysis applies equally well when using these “multi-move swaps” and I could analyze all clusters in this way if I wanted to, but the algorithm is slower in this mode. In general, in the interest of efficiency, I conduct all state-level analysis with single-move swaps, cluster-level VTD-level runs with multi-move swaps, and cluster-level geounit runs with single-move swaps, but additionally use multi-move swaps any time it enables the algorithm to generate more comparison maps.

Technical details of my implementation of these multi-moves are found in Appendix A. A related implementation detail for VTD splitting is also discussed there.

5.0.1 The *seats expected* metric for comparing districtings

As described in Section 4.2.1, my definition of optimized for partisanship involves comparing the partisanship of an enacted plan to the partisanship of comparison districtings produced from it by a sequence of random changes. Here I describe the *seats expected* metric of partisanship I use for this comparison throughout this report. In short, the seats expected metric for the districting is **the average number of seats Democrats would expect to win in the districting**, based on a uniform swing model with the historical voting data I use.

The *uniform swing* is a simple model frequently used to make predictions about the number of seats a party might win in an election, based on partisan voting data. Suppose, for example, that given data from the last North Carolina House election, we would like to predict how many seats Democrats will win in an upcoming House election (with the same districting), assuming that at a statewide level, we expect them to outperform by 1.5 percentage points their results from the last election.

A uniform swing would simply add 1.5 percentage points to Democrat performance in every district in data from the last election, and then evaluate how many seats would be won with these shifted voting outcomes.

When I am evaluating the partisanship of a comparison districting (to compare it to the enacted plan), I am interested in the number of seats we expect Democrats might win in the districting, given unknown shifts in partisan support. In particular, the metric I use is:

> How many seats, on average, would Democrats win in the given districting, if a random uniform swing is applied to the historical voting data being used?

As an example, let us consider the enacted Congressional plan, using the 2020 Attorney General election as a proxy for partisan voting patterns. Using these results as a direct proxy for future voting patterns, the enacted map would produce a 4:10 split of Democrat:Republican seats. If the Democrat vote share was increased by 1.68% in every district, the split would change to 5:9, and if it was increased by 3.05%, the split would rise to 6:8.

The random choice of my uniform swing is made from a normal distribution whose standard deviation is 4 percentage points, which is roughly the standard deviation of the swing in the past five North Carolina gubernatorial elections. Figure 1 visualizes the probabilities that this distribution assigns to the various seat splits which would arise from the enacted Congressional map under uniform swings of the 2020 Attorney General election:

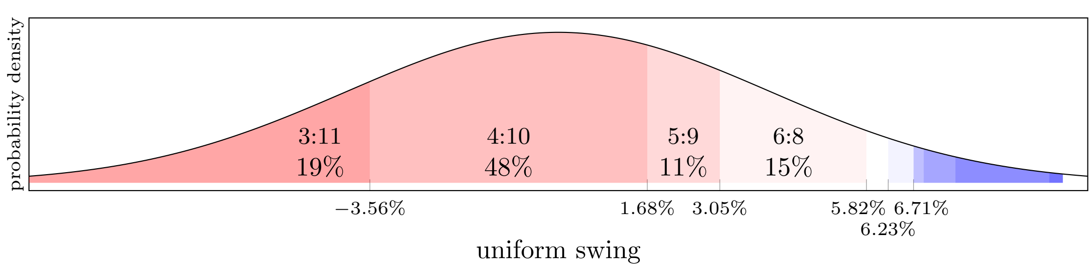


Figure 1: A normally distributed uniform swing applied to the enacted Congressional districting.

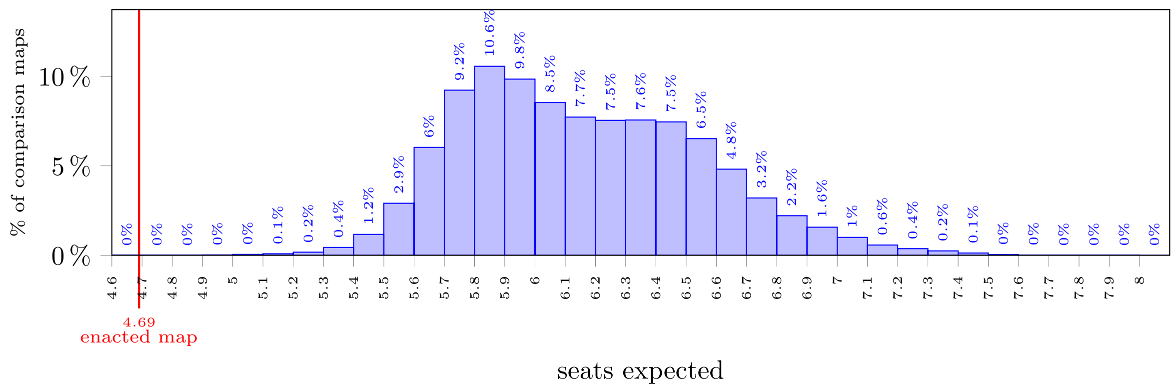
In particular, we can list the probability of any number of Democratic seats for the enacted Congressional plan according to this uniform swing model using the 2020 Attorney General race:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0.0%	0.0%	0.0%	19%	48%	11%	15%	1.3%	1.3%	0.1%	0.5%	1.2%	2.0%	0.1%	0.9%

The weighted average of these seat outcomes is computed as

$$.19 \times 3 + .48 \times 4 + .11 \times 5 + .15 \times 6 + .013 \times +.013 \times 8 + .001 \times 9 + .005 \times 10 + .012 \times 11 + .02 \times 12 + .001 \times 13 + .009 \times 14 = 4.69. \quad (1)$$

This "seats expected" number for the Congressional plan shows up in our analysis page for the Congressional districting (page 13), in a histogram we reproduce here for the purpose of illustration:



It is important to note that **my method does not evaluate the fairness of a districting by whether it produces a "small" or "large" number of seats for one party, or whether the uniform swing score calculated in this way is lower or higher than would be expected in a system of proportional representation**. Instead, this score is merely a metric used to **compare** one map to another. The only way these scores are used in my method is to evaluate which of two maps may be more advantageous to a particular political party, and when I find that a districting made extreme use of partisan consideration, it means that the enacted map is extreme outlier with respect to how optimized for partisanship it is **compared to the set of alternative comparison districtings of North Carolina** satisfying the districting criteria I impose.

5.0.2 Note on Population Deviation

My method does not simulate the results of hypothetical elections at the per-person level, and I do not enforce 1-person population deviation on Congressional districts. Instead, I use a cutoff 2%, as described above. I verify that the distinction between 1-person and 2% population deviation do not drive the results of my analysis in two ways.

First, in Section C, I show a run of my whole Congressional analysis exactly the same way but with a 1% population deviation constraint and obtain similar results. I also show a geounit-level analysis which operates at just 0.5% population deviation and still finds the enacted plan to be an extreme outlier.

Second, I analyze a coarse VTD-level version of the enacted map (itself with nearly 2% population deviation), and show that even this coarse version of the enacted map is an extreme outlier with respect to partisan bias, before small changes are made to it to produce the enacted 1-person-deviation map. This demonstrates that the coarse VTD-level "blueprint" for the map is an extreme outlier, optimized for partisan considerations, among alternative VTD-level maps with similar population deviation, even before the small changes used to achieve 1-person deviation are accounted for.

Finally, I note that by design, the seats-expected metric I use is not sensitive to the kinds of small changes that need to be made to districts to equalize population. This can already be seen by comparing the seats-expected metric for the enacted Congressional plan to the "VTD-level blueprint" version we analyze in Section C.8. As calculated above, the enacted map, with 1-person popluation deviation, scores 4.69 on the seats expected metric. The whole-VTD level blueprint, which has 1.8% population deviation, scores 4.70 by the same metric, as seen in the plot in C.8. This difference of 0.01 is much smaller than the sizes of differences in the seats-expected metric that are driving the results in my report.

5.1 A note on comparing results

For my cluster-by-cluster analysis of the House and Senate districtings, we will see that even among clusters for which we find that the enacted plan is an extreme outlier, there is quite a bit of variation from cluster to cluster for how extreme an outlier we find the enacted plan to be.

For example, in our second-level analysis of the Guilford county house districting, we find that it is among the most optimized-for-partisanship 00.000089% of all alternative districtings of the county satisfying our districting criteria, while for the Mecklenburg county districting, we find that it is among the most optimized-for-partisanship 5% of districtings.

Because it is tempting to compare results from cluster to cluster, it is important to emphasize that the mathematical results we employ in these findings are one-directional. In particular, while they imply that the Mecklenburg cluster is among the most optimized-for-partisanship 5% of districtings, they do *not* imply that it could not also be among the most optimized-for-partisanship 00.000089%.

What we know from my analysis is that we have extreme statistical certainty that the Guilford cluster districting is among the most optimized-for-partisanship 00.000089% of all districtings satisfying the criteria I consider, and we have extreme statistical certainty that the Mecklenburg cluster is among the most optimized-for-partisanship 5% of all districtings satisfying the criteria. The Mecklenburg cluster may be even more of an outlier, but my analysis does not address this latter question in either direction.

It should also be noted that it is natural to expect that my very conservative application of the districting criteria (discussed in Section 4.3.2) will affect some clusters more than others. In some clusters (e.g., Duplin/Wayne), it even prevents any comparison districtings from being generated by my algorithm at all. Of course, this should not seen as settling in either direction the question of whether the enacted map of the Duplin/Wayne cluster is gerrymandered.

6 Results of Analysis

The following pages show the results of my analysis for the enacted Congressional, state House, and state Senate districting plans.

Each page has the following components:

Comparison map examples

I show four maps in each case. The first map is the enacted map. The other three are examples of comparison maps used by by method. In each case, these maps are either the final map from runs 1, 2 and 3, or, from just the first run, the last map, the map from the halfway point of the run, and the run from the 25% point of the run.

Results

Under results I show a **table**, with an entry for each run conducted for the districting. The table shows the fraction of maps in that run that exhibited less partisan bias in favor of Republicans than the enacted map under evaluation. In particular, this is the fraction of maps for which the "seats expected" metric was higher than for the enacted map. For example, on the next page, we will see that in the first run, 99.9999947% of comparisons exhibited less partisan bias in favor of Republicans than the enacted plan.

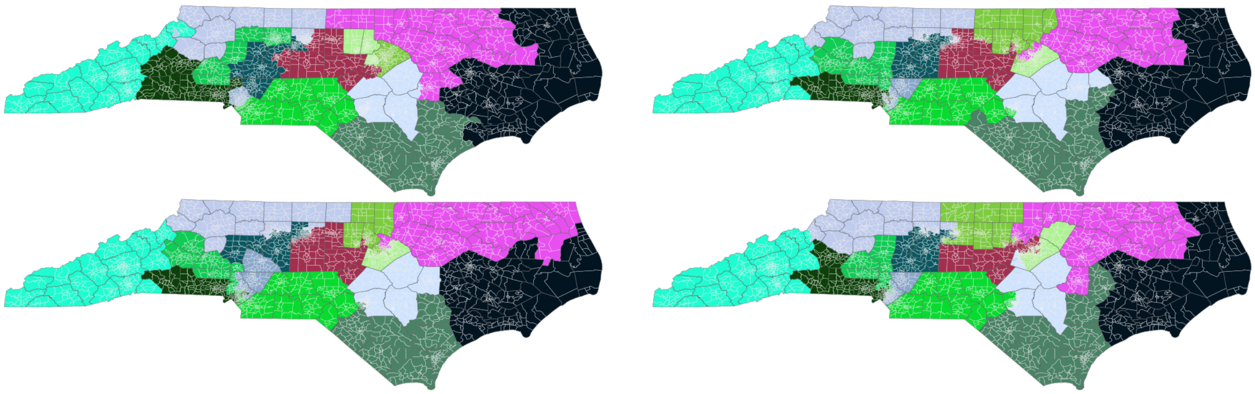
Below this table I show a **histogram** which plots the number of comparison maps whose "seats expected" value fell in various ranges. For example, on the next page, we see that 10.6% of comparison maps had a seats-expected value between 5.8 and 5.9. The histogram also shows the seats-expected value for the enacted map, which for the Congressional districting is 4.69. Note that the computation of this value 4.69 was illustrated earlier in Section 5.0.1. The same computation can be applied to every comparison map to build the histogram of resulting seats-expected values.

I present in each case a **First-level analysis**, which is simply a summary of the how the enacted map compares to the set of comparison districtings generated by my algorithm. For example, for the Congressional map, we will see that in every one of the 32 runs I conducted, 99.999968% of maps produced exhibited less partisan bias than the enacted map itself.

After this I present the **Second-level analysis**, which is a rigorous evaluation of how the enacted map compares to *all* alternative districtings of North Carolina satisfying the districting criteria I consider here. For example, for the Congressional districting as evaluated on the next page, we see that it is more optimized-for-partisanship than 99.999905% of *all* alternative districtings of North Carolina satisfying the criteria I impose as outlined in Section 4.3.1.

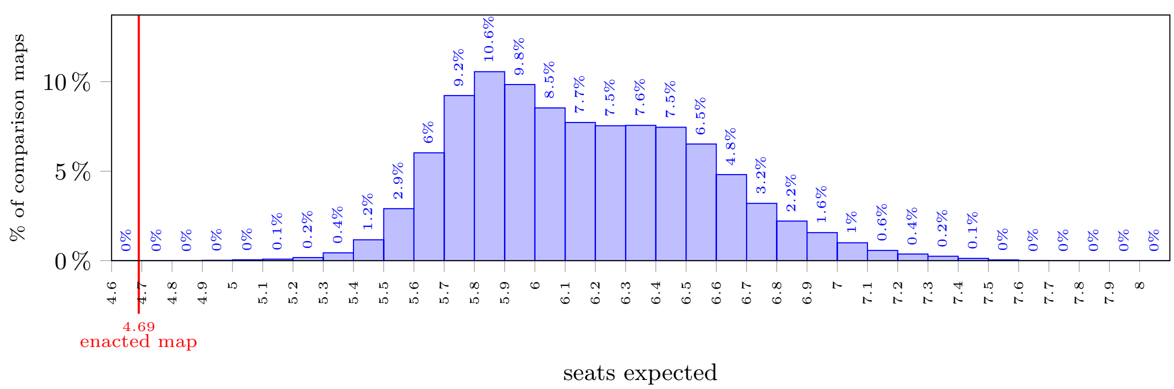
6.1 Congressional districting

6.1.1 Comparison map examples



6.1.2 Results

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999947%	9	99.9999909%	17	99.9999955%	25	99.999995%
2	99.999968%	10	99.99999966%	18	99.9999973%	26	99.9999961%
3	99.9999988%	11	99.9999943%	19	99.99999972%	27	99.99999977%
4	99.99999931%	12	99.999988%	20	99.9999999981%	28	99.99999979%
5	99.99999999927%	13	99.999988%	21	99.9999999962%	29	99.9999981%
6	99.9999959%	14	99.9999987%	22	99.99999919%	30	99.9999941%
7	99.99999984%	15	99.999996%	23	99.9999908%	31	99.99999901%
8	99.9999999947%	16	99.999985%	24	99.999981%	32	99.9999969%

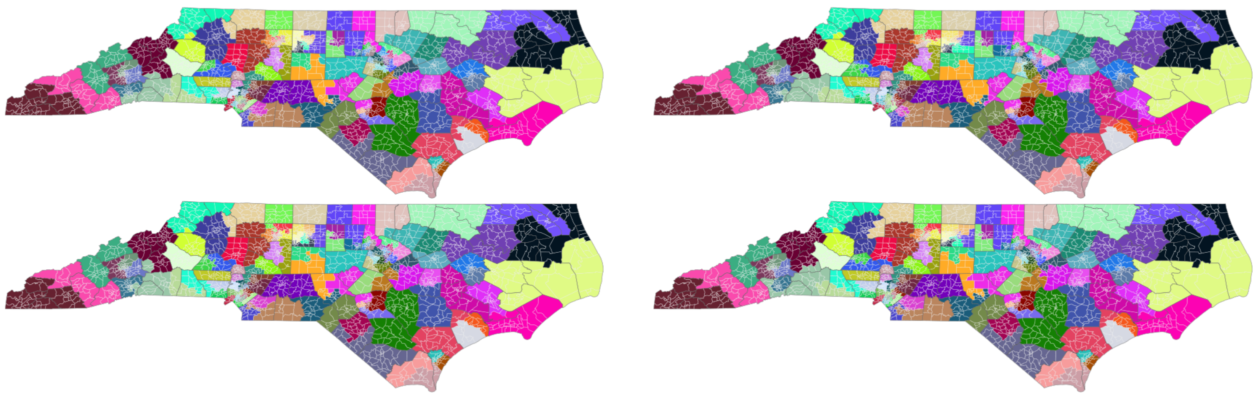


- **First level analysis:** In *every* run, the districting was in the most partisan 0.000031% of districtings (in other words, 99.999968% were less partisan, in *every* run).

- **Second level analysis:** My theorems imply that the enacted House districting is among the most optimized-for-partisanship 0.000094% of all alternative districtings of North Carolina satisfying my districting criteria (in other words, 99.999905% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.000031\%$.

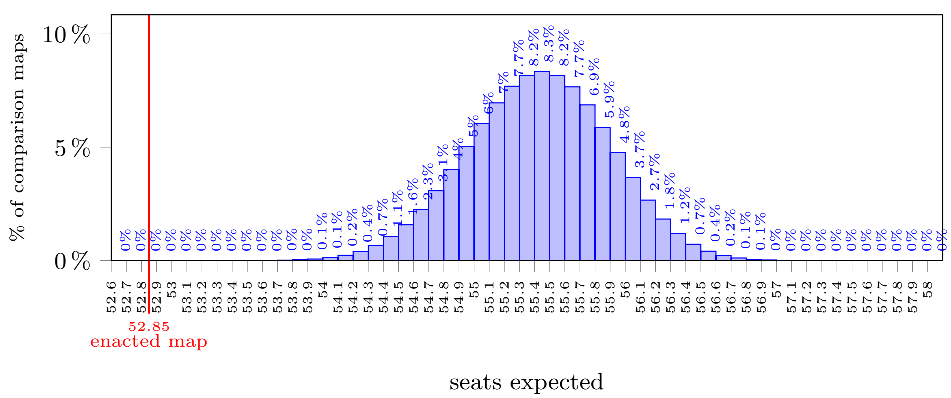
6.2 House districting

6.2.1 Comparison map examples



6.2.2 Results

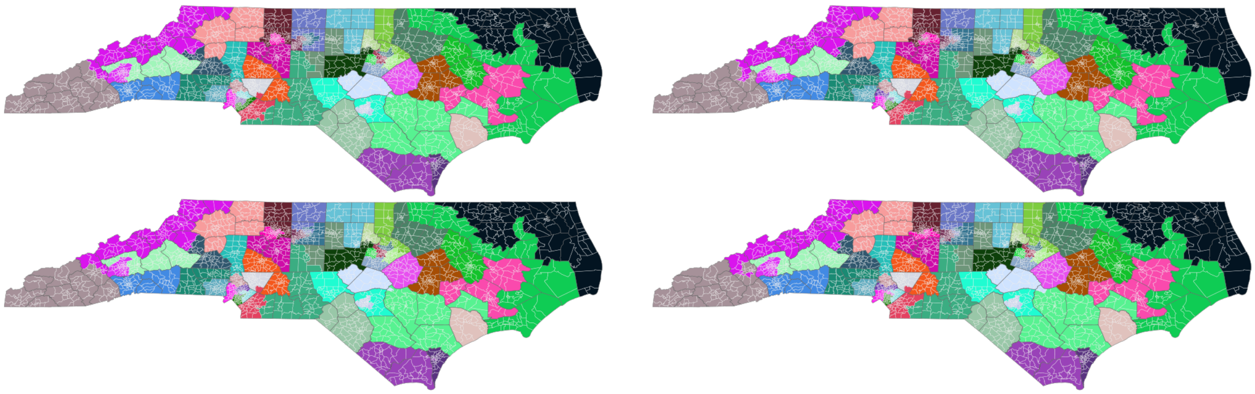
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.999999985%	9	99.99999957%	17	99.9999989%	25	99.9999989%
2	99.99999942%	10	99.99999904%	18	99.99999966%	26	99.9999918%
3	99.99999997%	11	99.9999984%	19	99.99999982%	27	99.99999984%
4	99.9999969%	12	99.9999986%	20	99.9999986%	28	99.9999988%
5	99.9999975%	13	99.99999989%	21	99.9999935%	29	99.99999987%
6	99.9999999959%	14	99.99999996%	22	99.9999999967%	30	99.99999908%
7	99.999999985%	15	99.9999984%	23	99.9999975%	31	99.9999966%
8	99.999999951%	16	99.99999954%	24	99.999999939%	32	99.999999939%



- **First level analysis:** In *every* run, the districting was in the most partisan 0.0000081% of districtings (in other words, 99.9999918% were less partisan, in *every* run).
- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.000024% of all alternative districtings of North Carolina satisfying my districting criteria (in other words, 99.999975% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.0000081\%$.

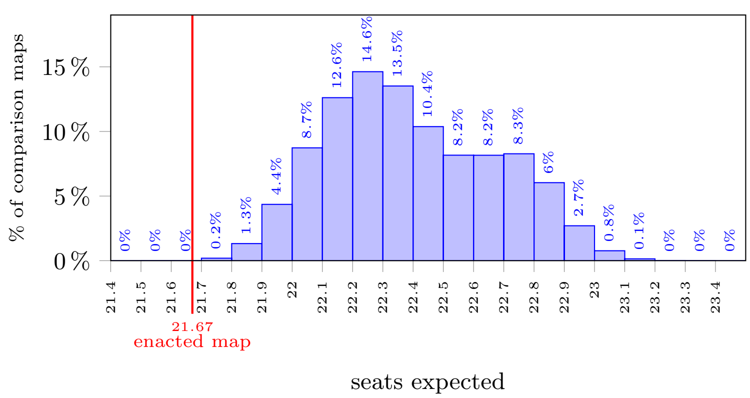
6.3 Senate districting

6.3.1 Comparison map examples



6.3.2 Results

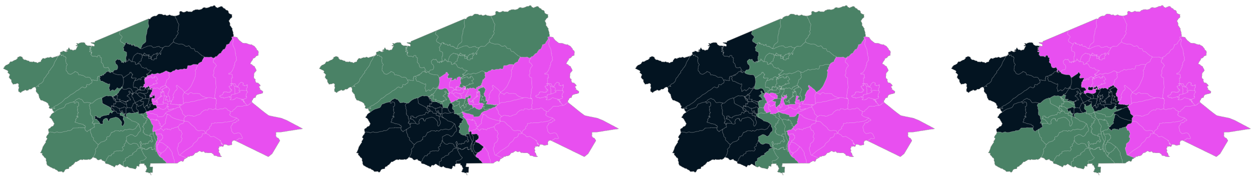
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.988%	9	99.9974%	17	99.9977%	25	99.998%
2	99.9988%	10	99.9958%	18	99.9987%	26	99.9948%
3	99.9938%	11	99.9985%	19	99.9988%	27	99.987%
4	99.9981%	12	99.9957%	20	99.978%	28	99.9988%
5	99.9929%	13	99.988%	21	99.9982%	29	99.9979%
6	99.9916%	14	99.989%	22	99.9978%	30	99.9981%
7	99.9957%	15	99.9974%	23	99.9976%	31	99.99914%
8	99.9973%	16	99.997%	24	99.9975%	32	99.9978%



- **First level analysis:** In *every* run, the districting was in the most partisan 0.021% of districtings (in other words, 99.978% were less partisan, in *every* run).
- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.065% of all alternative districtings of North Carolina satisfying my districting criteria (in other words, 99.934% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.021\%$.

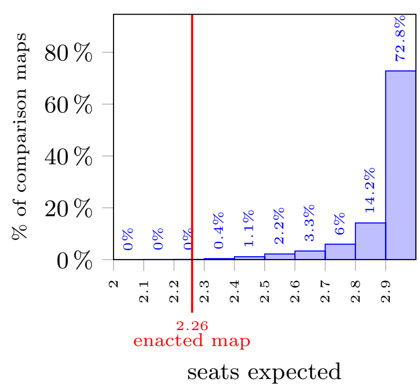
6.4 House Cluster: Buncombe

6.4.1 Comparison map examples



6.4.2 Results

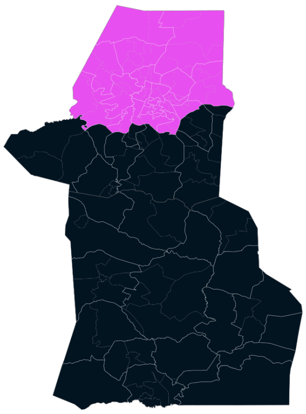
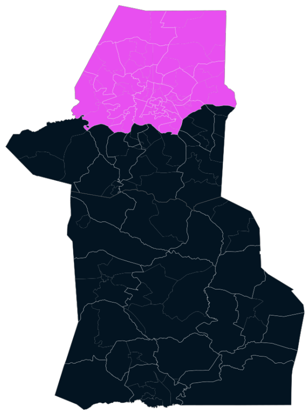
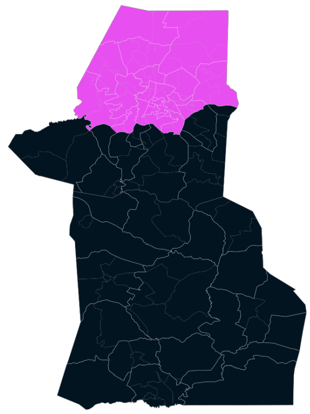
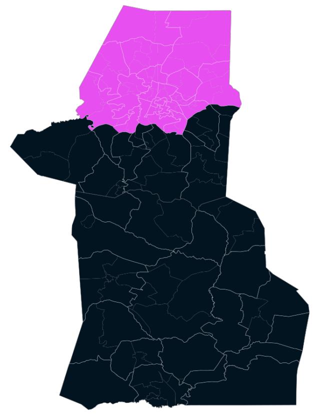
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.979%	9	99.979%	17	99.979%	25	99.98%
2	99.98%	10	99.98%	18	99.979%	26	99.979%
3	99.98%	11	99.98%	19	99.98%	27	99.979%
4	99.98%	12	99.98%	20	99.98%	28	99.98%
5	99.98%	13	99.98%	21	99.98%	29	99.98%
6	99.979%	14	99.98%	22	99.98%	30	99.98%
7	99.98%	15	99.98%	23	99.98%	31	99.979%
8	99.979%	16	99.98%	24	99.98%	32	99.979%



- **First level analysis:** In *every* run, the districting was in the most partisan 0.020% of districtings (in other words, 99.979% were less partisan, in *every* run).
- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.061% of all alternative districtings satisfying my districting criteria (in other words, 99.938% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.020\%$.

6.5 House Cluster:Duplin/Wayne

6.5.1 Comparison map examples

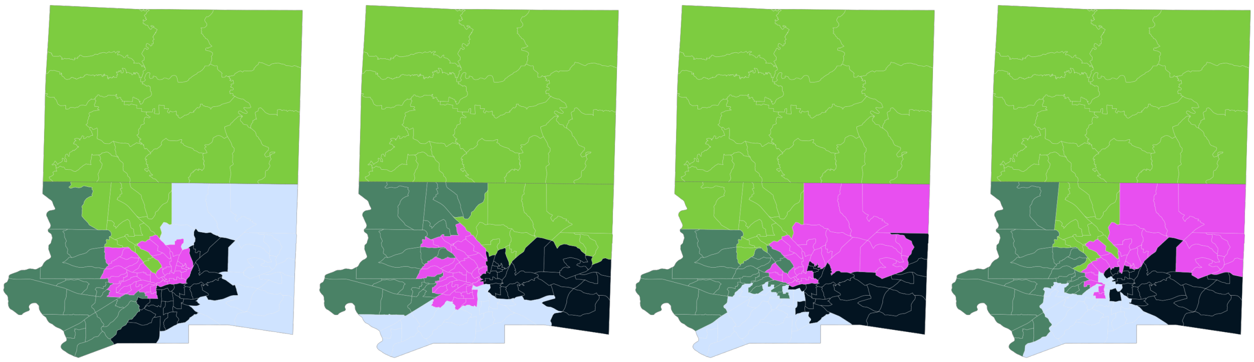





6.5.2 Results

- For this cluster, my conservative approach (as discussed in Section 4.3.2) does not allow my algorithm to generate any comparison maps other than the map itself.

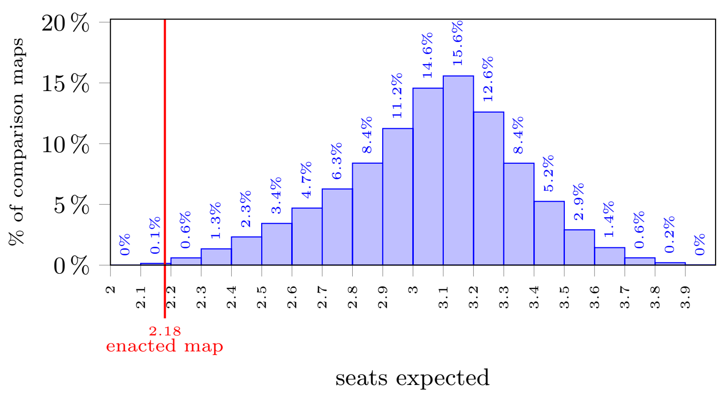
6.6 House Cluster: Forsyth-Stokes

6.6.1 Comparison map examples



6.6.2 Results

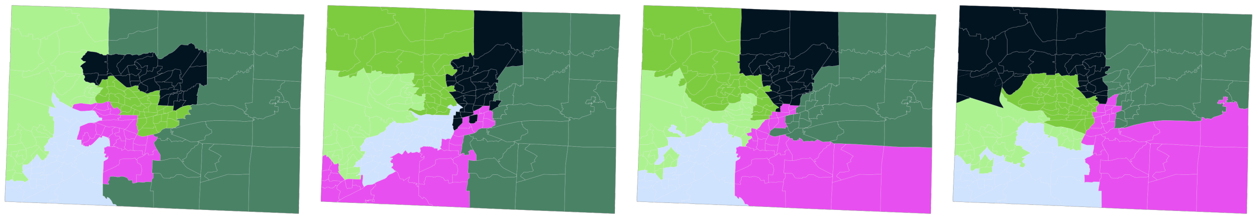
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.913%	9	99.912%	17	99.915%	25	99.914%
2	99.914%	10	99.914%	18	99.914%	26	99.913%
3	99.917%	11	99.912%	19	99.916%	27	99.914%
4	99.916%	12	99.912%	20	99.914%	28	99.912%
5	99.913%	13	99.914%	21	99.913%	29	99.915%
6	99.913%	14	99.914%	22	99.914%	30	99.914%
7	99.913%	15	99.912%	23	99.914%	31	99.917%
8	99.913%	16	99.916%	24	99.915%	32	99.915%



- **First level analysis:** In *every* run, the districting was in the most partisan 0.087% of districtings (in other words, 99.912% were less partisan, in *every* run).
- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.26% of all alternative districtings satisfying my districting criteria (in other words, 99.73% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.087\%$.

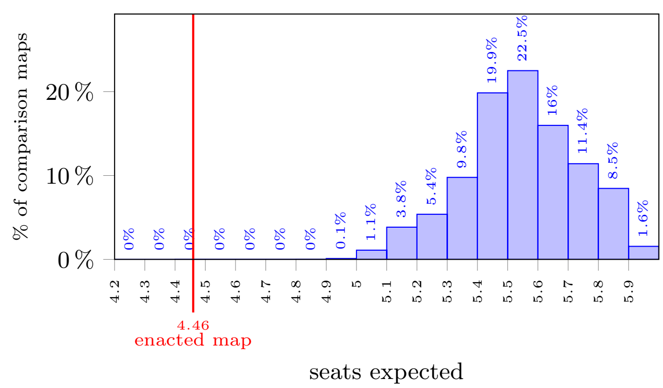
6.7 House Cluster: Guilford

6.7.1 Comparison map examples



6.7.2 Results

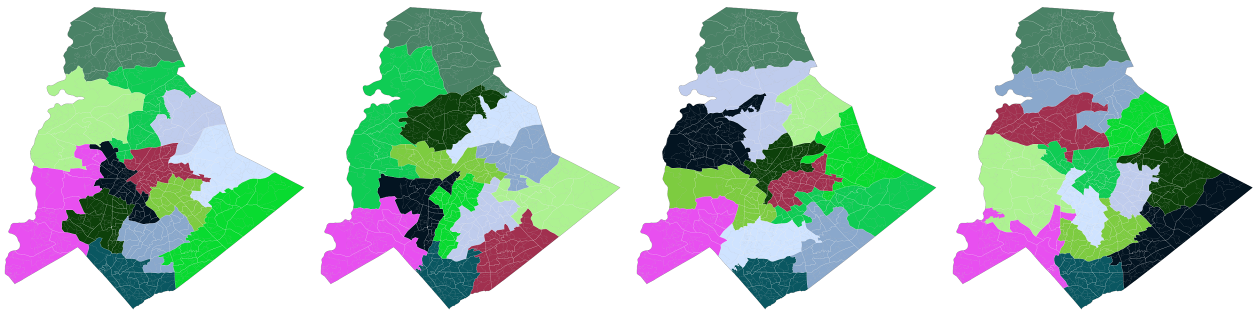
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.999989%	9	99.999982%	17	99.999979%	25	99.999972%
2	99.999982%	10	99.999979%	18	99.999978%	26	99.999979%
3	99.999972%	11	99.999978%	19	99.999981%	27	99.999978%
4	99.999986%	12	99.999981%	20	99.999984%	28	99.999979%
5	99.999975%	13	99.999986%	21	99.999983%	29	99.999982%
6	99.999982%	14	99.99998%	22	99.999979%	30	99.999982%
7	99.999981%	15	99.99997%	23	99.999983%	31	99.999982%
8	99.999982%	16	99.999976%	24	99.999981%	32	99.999984%



- **First level analysis:** In *every* run, the districting was in the most partisan 0.000029% of districtings (in other words, 99.99997% were less partisan, in *every* run).
- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.000089% of all alternative districtings satisfying my districting criteria (in other words, 99.99991% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.000029\%$.

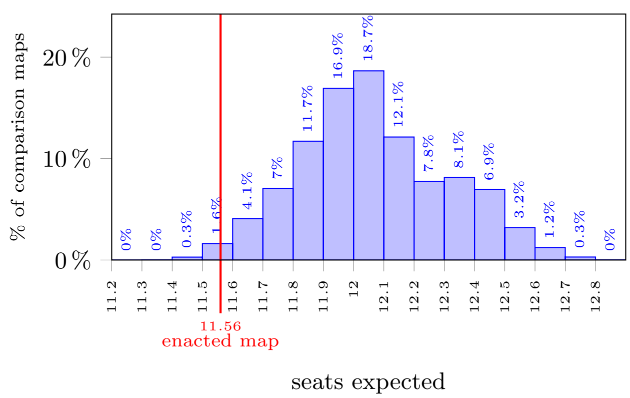
6.8 House Cluster: Mecklenburg

6.8.1 Comparison map examples



6.8.2 Results

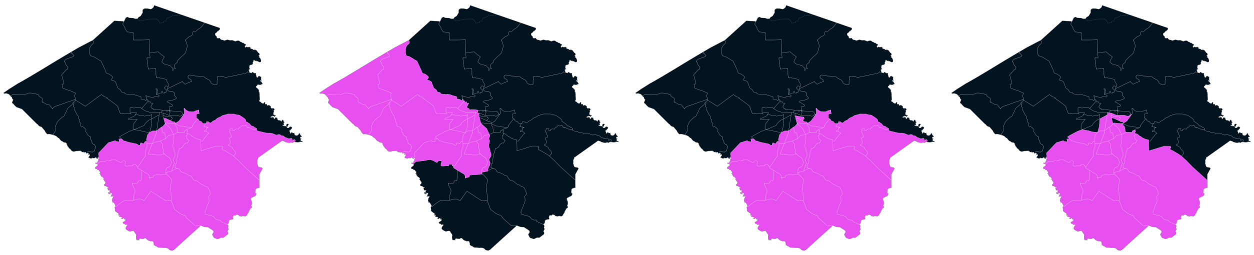
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	98.7%	9	98.6%	17	98.4%	25	98.9%
2	99.36%	10	99.15%	18	99.%	26	98.3%
3	98.7%	11	98.7%	19	98.4%	27	98.8%
4	99.14%	12	99.17%	20	99.17%	28	98.5%
5	98.4%	13	99.05%	21	98.8%	29	99.08%
6	99.33%	14	99.02%	22	98.9%	30	98.9%
7	98.5%	15	99.%	23	98.9%	31	99.12%
8	98.9%	16	99.17%	24	98.9%	32	99.2%



- **First level analysis:** In *every* run, the districting was in the most partisan 1.7% of districtings (in other words, 98.3% were less partisan, in *every* run).
- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 5.0% of all alternative districtings satisfying my districting criteria (in other words, 95.0% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 1.7\%$.

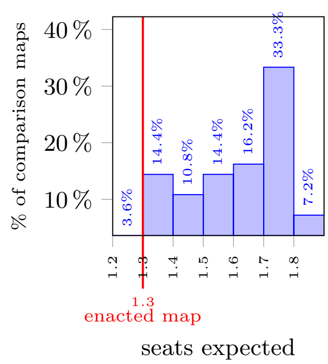
6.9 House Cluster: Pitt

6.9.1 Comparison map examples



6.9.2 Results

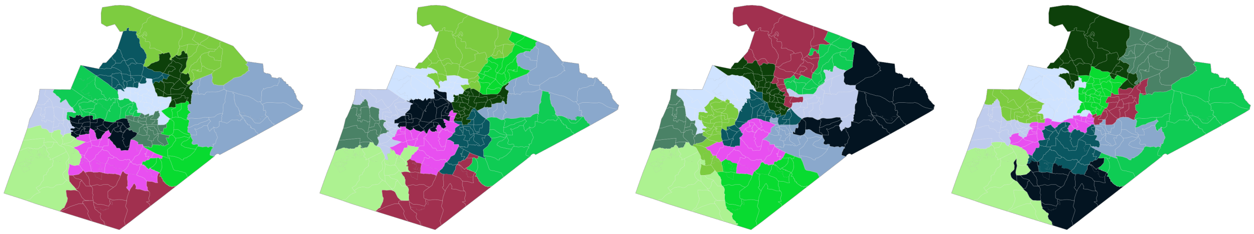
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	96.3%	9	96.4%	17	96.3%	25	96.4%
2	96.3%	10	96.3%	18	96.3%	26	96.3%
3	96.4%	11	96.4%	19	96.3%	27	96.4%
4	96.4%	12	96.4%	20	96.3%	28	96.3%
5	96.4%	13	96.4%	21	96.3%	29	96.4%
6	96.3%	14	96.3%	22	96.4%	30	96.3%
7	96.3%	15	96.3%	23	96.4%	31	96.4%
8	96.3%	16	96.4%	24	96.4%	32	96.4%



- **First level analysis:** In *every* run, the districting was in the most partisan 3.6% of districtings (in other words, 96.3% were less partisan, in *every* run).
- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 11% of all alternative districtings satisfying my districting criteria (in other words, 89.1% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 3.6\%$.

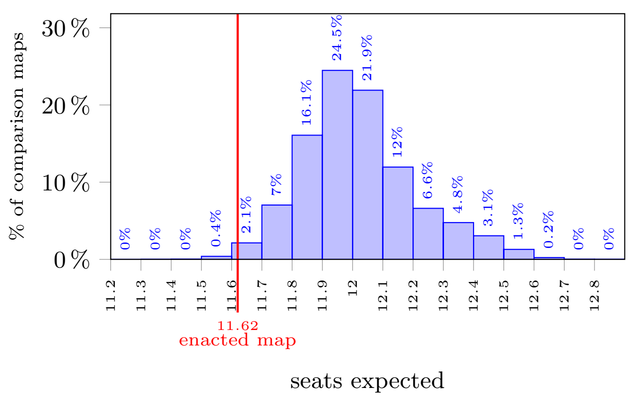
6.10 House Cluster: Wake

6.10.1 Comparison map examples



6.10.2 Results

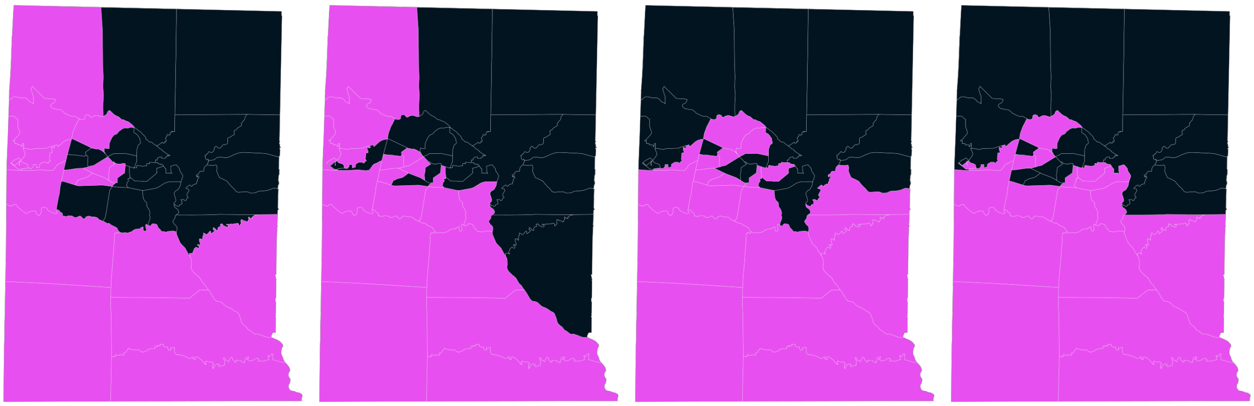
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.38%	9	99.34%	17	99.37%	25	99.35%
2	99.33%	10	99.35%	18	99.36%	26	99.36%
3	99.34%	11	99.33%	19	99.33%	27	99.34%
4	99.32%	12	99.34%	20	99.35%	28	99.33%
5	99.35%	13	99.34%	21	99.33%	29	99.35%
6	99.33%	14	99.27%	22	99.31%	30	99.36%
7	99.34%	15	99.34%	23	99.32%	31	99.36%
8	99.34%	16	99.36%	24	99.35%	32	99.35%



- **First level analysis:** In *every* run, the districting was in the most partisan 0.72% of districtings (in other words, 99.27% were less partisan, in *every* run).
- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 2.2% of all alternative districtings satisfying my districting criteria (in other words, 97.8% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.72\%$.

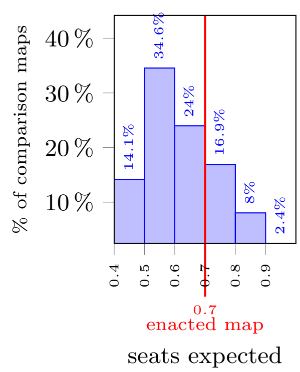
6.11 House Cluster: Alamance

6.11.1 Comparison map examples



6.11.2 Results

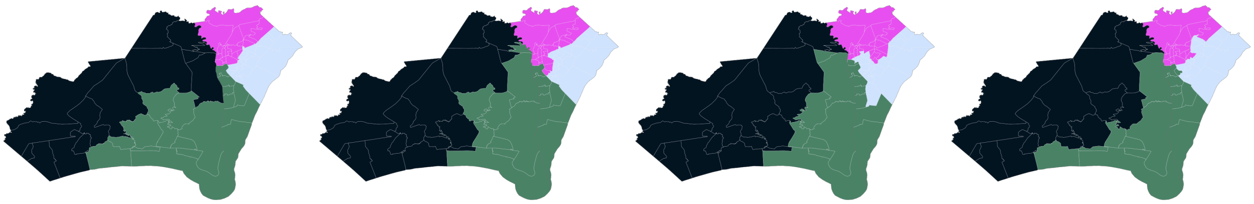
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	26.3%	9	26.4%	17	26.3%	25	26.4%
2	26.3%	10	26.3%	18	26.4%	26	26.3%
3	26.3%	11	26.3%	19	26.3%	27	26.3%
4	26.4%	12	26.3%	20	26.3%	28	26.3%
5	26.4%	13	26.4%	21	26.4%	29	26.3%
6	26.3%	14	26.3%	22	26.4%	30	26.4%
7	26.4%	15	26.3%	23	26.3%	31	26.3%
8	26.4%	16	26.4%	24	26.4%	32	26.4%



- **First level analysis:** In *every* run, the districting was in the most partisan 74% of districtings (in other words, 26.3% were less partisan, in *every* run).
- **Second level analysis:** The enacted map is not unusual enough in the first-level analysis to enable a statistically significant second-level analysis of this cluster.

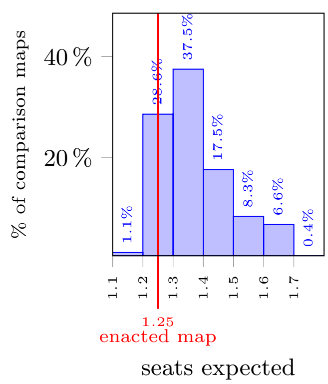
6.12 House Cluster: Brunswick/New Hanover

6.12.1 Comparison map examples



6.12.2 Results

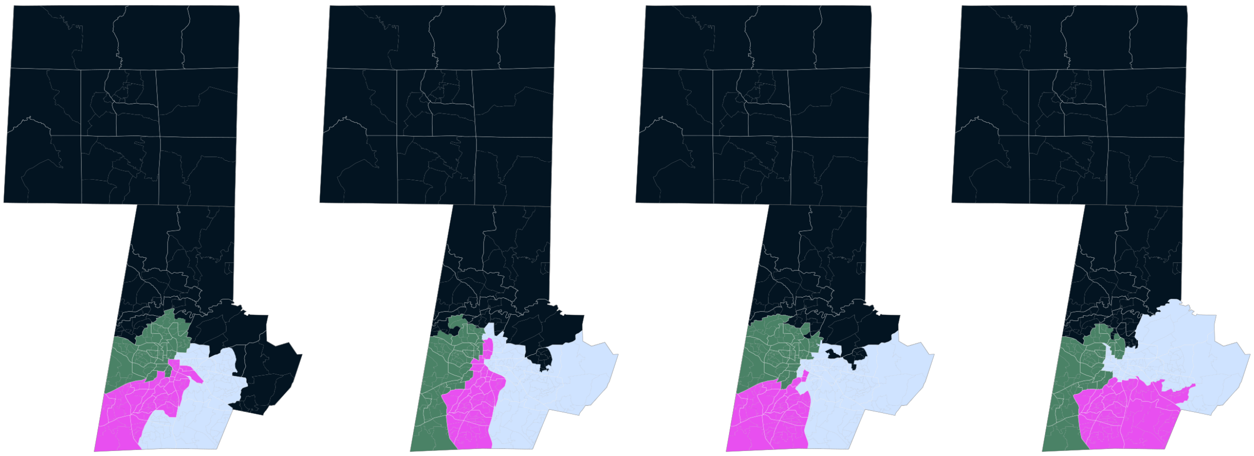
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	89.4%	9	89.5%	17	89.5%	25	89.5%
2	89.4%	10	89.5%	18	89.4%	26	89.5%
3	89.5%	11	89.5%	19	89.5%	27	89.4%
4	89.4%	12	89.4%	20	89.4%	28	89.5%
5	89.4%	13	89.5%	21	89.5%	29	89.5%
6	89.5%	14	89.6%	22	89.5%	30	89.4%
7	89.4%	15	89.5%	23	89.5%	31	89.5%
8	89.5%	16	89.4%	24	89.4%	32	89.5%



- **First level analysis:** In *every* run, the districting was in the most partisan 11% of districtings (in other words, 89.4% were less partisan, in *every* run).
- **Second level analysis:** The enacted map is not unusual enough in the first-level analysis to enable a statistically significant second-level analysis of this cluster.

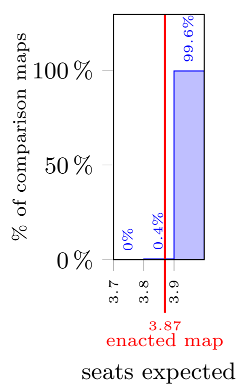
6.13 House Cluster: Durham/Person

6.13.1 Comparison map examples



6.13.2 Results

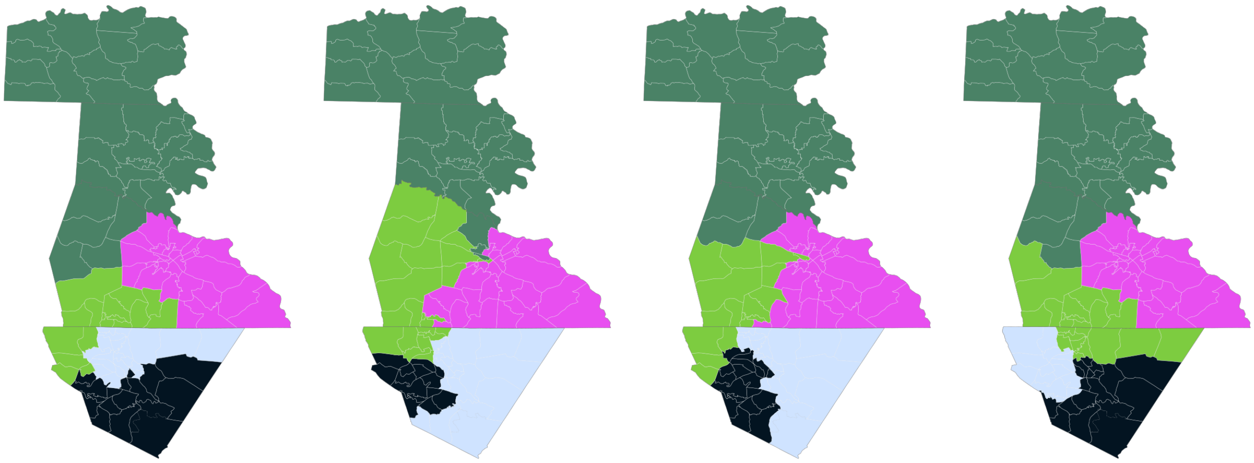
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.936%	9	99.935%	17	99.938%	25	99.935%
2	99.933%	10	99.937%	18	99.937%	26	99.933%
3	99.937%	11	99.94%	19	99.934%	27	99.939%
4	99.932%	12	99.933%	20	99.934%	28	99.936%
5	99.933%	13	99.936%	21	99.936%	29	99.937%
6	99.936%	14	99.935%	22	99.938%	30	99.933%
7	99.937%	15	99.933%	23	99.937%	31	99.94%
8	99.936%	16	99.936%	24	99.934%	32	99.934%



- **First level analysis:** In *every* run, the districting was in the most partisan 0.067% of districtings (in other words, 99.932% were less partisan, in *every* run).
- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.20% of all alternative districtings satisfying my districting criteria (in other words, 99.79% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.067\%$.

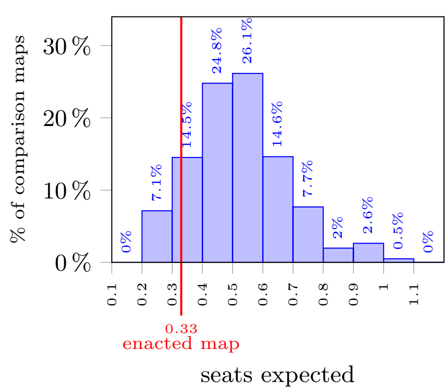
6.14 House Cluster: Cabarrus/Davie/Rowan/Yadkin

6.14.1 Comparison map examples



6.14.2 Results

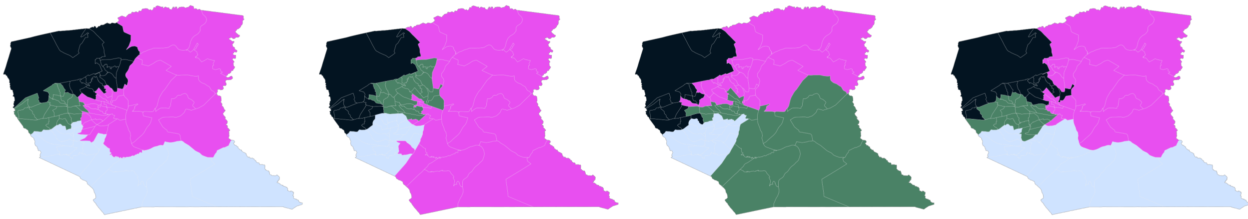
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	89.0%	9	90.0%	17	88.5%	25	89.9%
2	90.0%	10	88.9%	18	89.0%	26	88.6%
3	90.1%	11	88.7%	19	89.4%	27	89.9%
4	88.4%	12	89.8%	20	89.3%	28	88.9%
5	89.7%	13	89.4%	21	92.8%	29	89.5%
6	88.6%	14	89.2%	22	89.1%	30	87.7%
7	89.5%	15	88.8%	23	89.1%	31	90.2%
8	90.0%	16	90.0%	24	88.7%	32	90.4%



- **First level analysis:** In *every* run, the districting was in the most partisan 12% of districtings (in other words, 87.7% were less partisan, in *every* run).
- **Second level analysis:** The enacted map is not unusual enough in the first-level analysis to enable a statistically significant second-level analysis of this cluster.

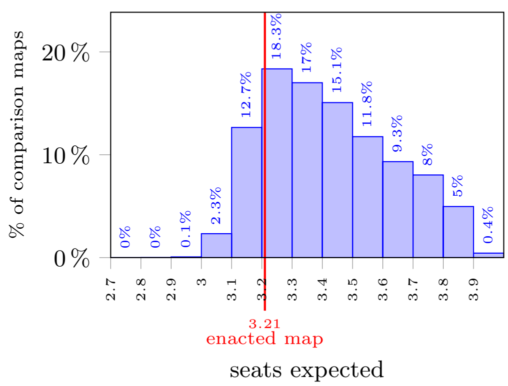
6.15 House Cluster: Cumberland

6.15.1 Comparison map examples



6.15.2 Results

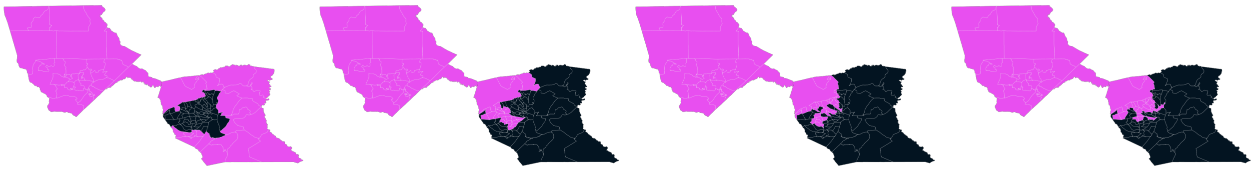
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	83.6%	9	83.8%	17	83.8%	25	84.0%
2	83.7%	10	83.9%	18	83.6%	26	83.5%
3	83.8%	11	83.8%	19	83.7%	27	83.8%
4	83.7%	12	83.6%	20	83.7%	28	83.8%
5	83.6%	13	83.7%	21	84.0%	29	83.7%
6	83.7%	14	83.6%	22	83.9%	30	83.6%
7	83.5%	15	83.8%	23	83.7%	31	83.9%
8	83.7%	16	83.8%	24	83.6%	32	83.9%



- **First level analysis:** In *every* run, the districting was in the most partisan 16% of districtings (in other words, 83.5% were less partisan, in *every* run).
- **Second level analysis:** The enacted map is not unusual enough in the first-level analysis to enable a statistically significant second-level analysis of this cluster.

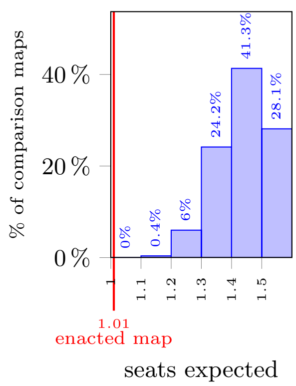
6.16 Senate Cluster: Cumberland Moore

6.16.1 Comparison map examples



6.16.2 Results

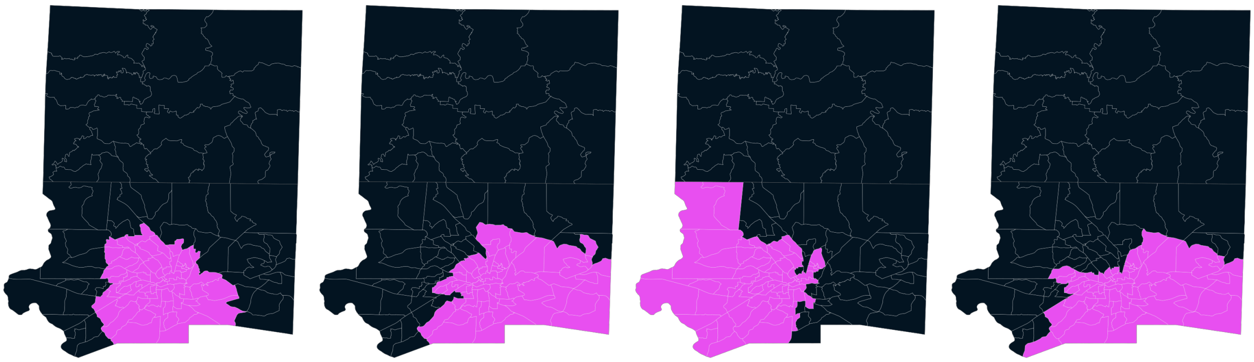
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999968%	9	99.9999962%	17	99.9999963%	25	99.9999954%
2	99.9999961%	10	99.9999965%	18	99.9999969%	26	99.9999955%
3	99.999998%	11	99.9999954%	19	99.9999967%	27	99.999997%
4	99.9999953%	12	99.9999961%	20	99.9999969%	28	99.9999952%
5	99.9999969%	13	99.9999957%	21	99.9999971%	29	99.9999959%
6	99.9999969%	14	99.9999949%	22	99.9999961%	30	99.9999956%
7	99.9999966%	15	99.9999964%	23	99.9999961%	31	99.9999961%
8	99.9999966%	16	99.9999959%	24	99.9999977%	32	99.9999965%



- **First level analysis:** In *every* run, the districting was in the most partisan 0.0000050% of districtings (in other words, 99.9999949% were less partisan, in *every* run).
- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.000015% of all alternative districtings satisfying my districting criteria (in other words, 99.999984% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.0000050\%$.

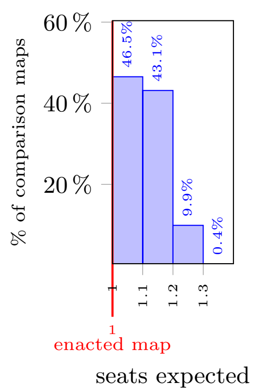
6.17 Senate Cluster: Forsyth-Stokes

6.17.1 Comparison map examples



6.17.2 Results

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9983%	9	99.9983%	17	99.9983%	25	99.9983%
2	99.9984%	10	99.9984%	18	99.9984%	26	99.9983%
3	99.9982%	11	99.9983%	19	99.9984%	27	99.9983%
4	99.9982%	12	99.9984%	20	99.9983%	28	99.9984%
5	99.9983%	13	99.9983%	21	99.9983%	29	99.9983%
6	99.9984%	14	99.9983%	22	99.9983%	30	99.9984%
7	99.9984%	15	99.9983%	23	99.9983%	31	99.9984%
8	99.9984%	16	99.9984%	24	99.9984%	32	99.9983%

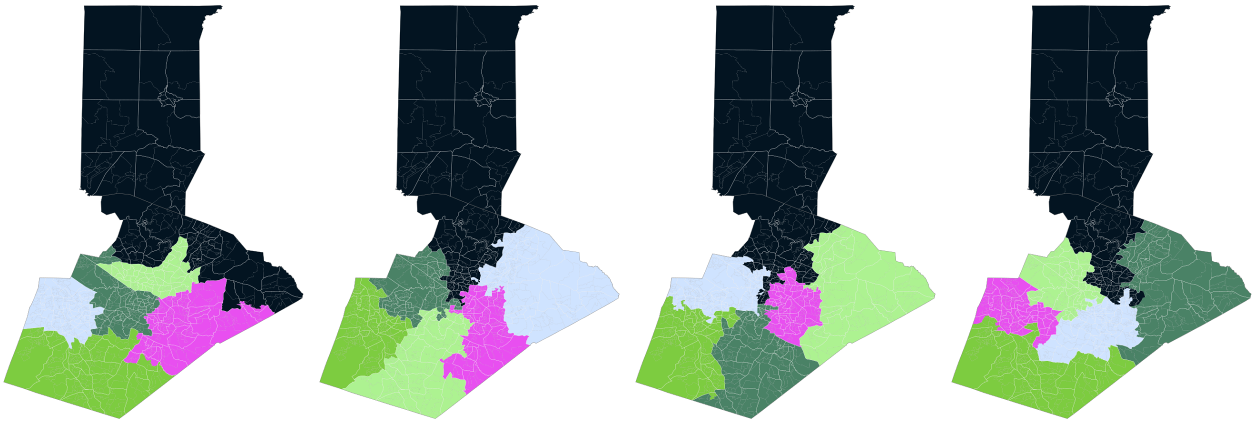


- **First level analysis:** In *every* run, the districting was in the most partisan 0.0016% of districtings (in other words, 99.9983% were less partisan, in *every* run).

- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.0051% of all alternative districtings satisfying my districting criteria (in other words, 99.9947% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.0016\%$.

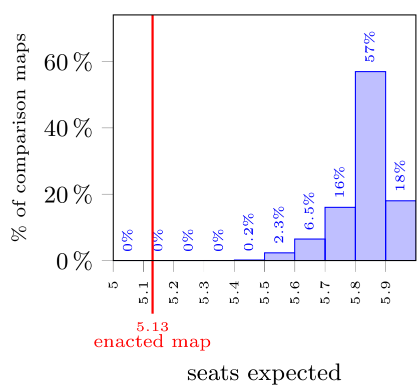
6.18 Senate Cluster: Granville-Wake

6.18.1 Comparison map examples



6.18.2 Results

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.99999934%	9	99.99999921%	17	99.99999999936%	25	99.9999971%
2	99.9999984%	10	99.99999999936%	18	99.99999913%	26	99.9999975%
3	99.99999917%	11	99.99999966%	19	99.9999967%	27	99.99999909%
4	99.99999999945%	12	99.9999979%	20	99.99999963%	28	99.999989%
5	99.99999974%	13	99.9999989%	21	99.9999999984%	29	99.99999999954%
6	99.999999939%	14	99.9999976%	22	99.99999948%	30	99.9999968%
7	99.9999999982%	15	99.9999947%	23	99.9999984%	31	99.99999999945%
8	99.9999995%	16	99.99999969%	24	99.99999967%	32	99.99999971%

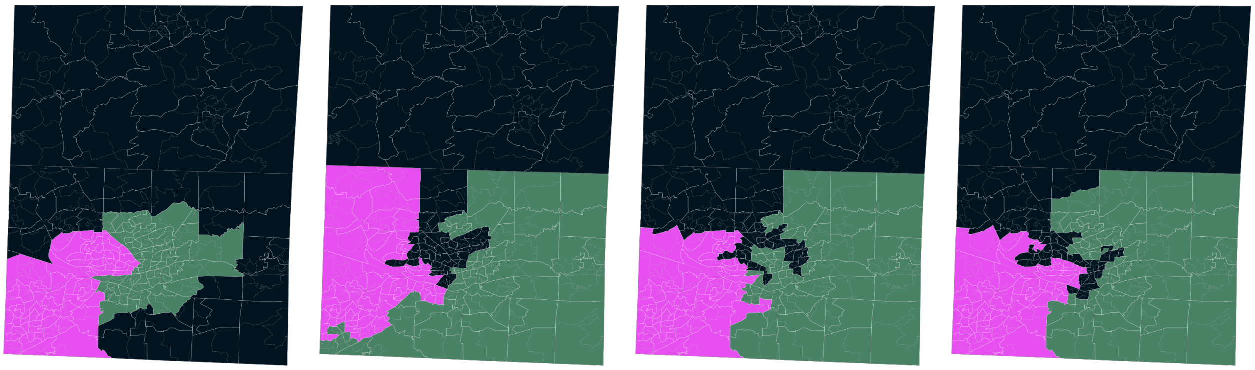


- **First level analysis:** In *every* run, the districting was in the most partisan 0.000010% of districtings (in other words, 99.999989% were less partisan, in *every* run).

- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.000030% of all alternative districtings satisfying my districting criteria (in other words, 99.999969% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.000010\%$.

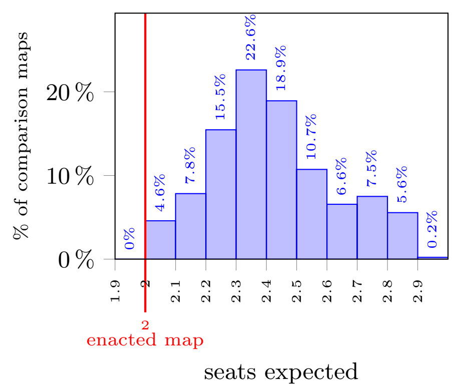
6.19 Senate Cluster: Guilford-Rockingham

6.19.1 Comparison map examples



6.19.2 Results

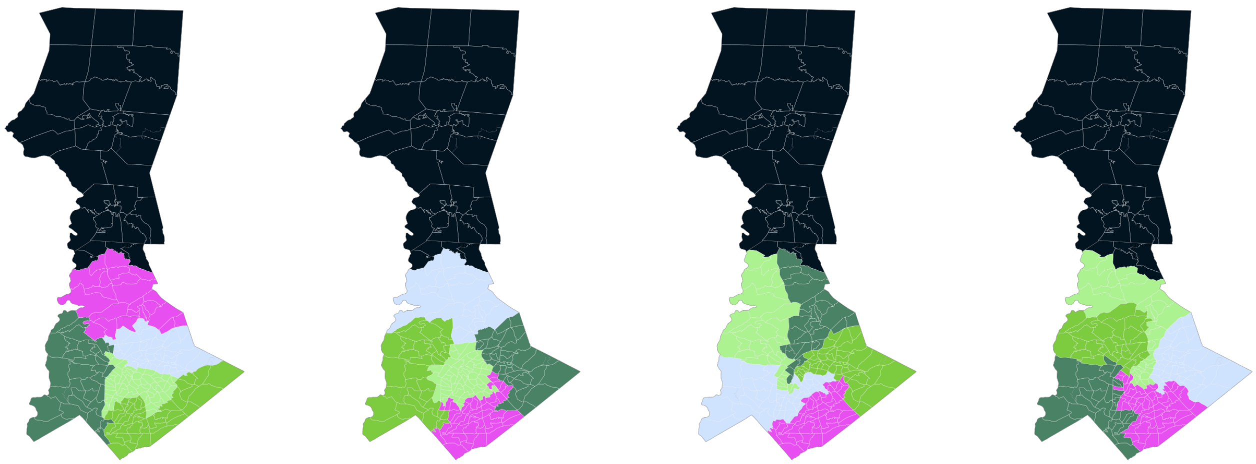
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999979%	9	99.9999971%	17	99.999989%	25	99.999984%
2	99.999975%	10	99.999999976%	18	99.9999929%	26	99.99999949%
3	99.9999991%	11	99.9999944%	19	99.999988%	27	99.999967%
4	99.999984%	12	99.99998%	20	99.99998%	28	99.999995%
5	99.999976%	13	99.9999978%	21	99.99996%	29	99.999957%
6	99.9999922%	14	99.999978%	22	99.999979%	30	99.9999999957%
7	99.9999997%	15	99.999986%	23	99.9999964%	31	99.9999935%
8	99.999967%	16	99.9999939%	24	99.999983%	32	99.9999984%



- **First level analysis:** In *every* run, the districting was in the most partisan 0.000042% of districtings (in other words, 99.999957% were less partisan, in *every* run).
- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.00012% of all alternative districtings satisfying my districting criteria (in other words, 99.99987% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.000042\%$.

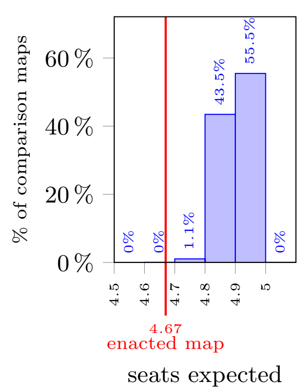
6.20 Senate Cluster: Iredell-Mecklenburg

6.20.1 Comparison map examples



6.20.2 Results

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9981%	9	99.9983%	17	99.9982%	25	99.9982%
2	99.9982%	10	99.9983%	18	99.9982%	26	99.9983%
3	99.9982%	11	99.9981%	19	99.9981%	27	99.9981%
4	99.9982%	12	99.9982%	20	99.9982%	28	99.9982%
5	99.9981%	13	99.9982%	21	99.9982%	29	99.9982%
6	99.9983%	14	99.9982%	22	99.9982%	30	99.9982%
7	99.9982%	15	99.9982%	23	99.9982%	31	99.9982%
8	99.9982%	16	99.9982%	24	99.9982%	32	99.9981%



- **First level analysis:** In *every* run, the districting was in the most partisan 0.0019% of districtings (in other words, 99.998% were less partisan, in *every* run).
- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.0057% of all alternative districtings satisfying my districting criteria (in other words, 99.9943% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.0019\%$.

7 Seat preservation analyses

In this section I present analyses of clusters for which my main analysis does not achieve high confidence of gerrymandering with respect to the seats-expected metric. These are the districtings in the following House clusters:

- Alamance
- Brunswick/New Hanover
- Cabarrus/Davie/Rowan/Yadkin
- Cumberland

Note that the motivation for the seat-expected metric is to detect partisan gerrymandering aimed at maximizing the expected total number of seats belonging to one party in a representative body (Congress, the North Carolina house, or the North Carolina senate). But there may be other conceivable partisan goals, such as facilitating the re-election of particular representatives in particular districts, which may be orthogonal to or (at least not perfectly correlated with) the goal of maximizing expected representation from one party, and thus which would not be detected by the seats-expected metric.

The metric I use in this section to re-analyze these districtings is the *wave threshold* for a particular seat count. In particular, for a given number of seats x, the wave threshold for x is the smallest uniform swing which can be applied to election data (here, the 2020 Attorney General race) which would result in $x + 1$ Democratic seats. Put differently, this is the threshold such that for any smaller uniform swing, the Democrats will win at most x seats. Referring back to Figure 1, we see that for the enacted Congressional districting of North Carolina, the wave thresholds for $x = 3, 4, 5$, and 6 are -3.56%, 1.68%, 3.05%, and 5.82%, respectively. In particular, even in an election in which voter patterns mirror the 2020 Attorney General race **with all Democratic vote shares increased by an additional** 5.81 **percentage points**, the enacted Congressional districting would still produce only 6 Democrat representatives.

The wave threshold metric can capture partisan goals which may be washed out in the seats-expected metric. For example, if a 5-district cluster is proposed to be districted to optimize the chance that three Republican incumbents all can save their seats, this may or may not result in an increase in the seats-expected metric (for example, if the alternative was to have 4 lean-Republican competitive districts, the extent of the lean would determine how the proposed and alternative districtings would compare under the seats expected metric). But such a plan would be expected to stand out as being highly unusual with respect to the wave threshold for 2 Democratic seats, as it would be an extreme outlier with respect to how difficult it would be for Democrats to capture more than 2 seats in the cluster.

All wave-threshold histograms are shown with red bars, to visually distinguish them from the seats-expected histograms shown elsewhere in the report. Note that unlike for the seats-expected histograms, a Republican bias in the enacted map with respect to a particular wave threshold is indicated by the enacted map showing as an outlier on the righthand side of the plot.

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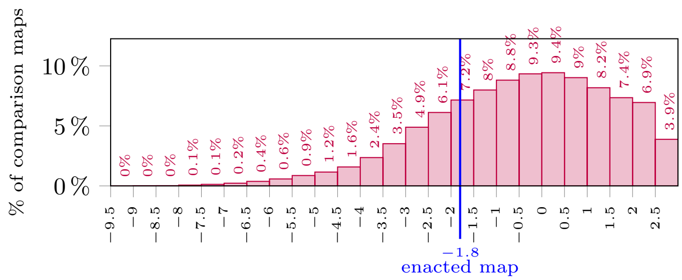
7.1 Alamance

The comparison maps generated by my algorithm were similar to the enacted map with respect to their wave threshold for both possible seat values (results here shown for the wave threshold for 0 seats):

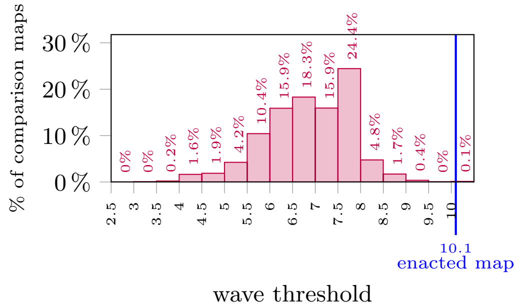
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	25.2%	9	25.2%	17	25.1%	25	25.2%
2	25.2%	10	25.0%	18	25.1%	26	25.2%
3	25.2%	11	25.1%	19	25.1%	27	25.2%
4	25.2%	12	25.2%	20	25.1%	28	25.2%
5	25.3%	13	25.2%	21	25.3%	29	25.3%
6	25.2%	14	25.2%	22	25.2%	30	25.2%
7	25.2%	15	25.1%	23	25.3%	31	25.3%
8	25.2%	16	25.2%	24	25.2%	32	25.2%



7.2 Brunswick/New Hanover

Despite the fact that my algorithm did not detect large differences between the enacted districting and comparison districtings of this cluster, the enacted map is an extreme outlier among the comparison maps generated by my algorithm with respect to the wave threshold for two seats. In particuliar, for the enacted map in this cluster, Democratic performance could increase by 10.1 percentage points in every district without Democrats capturing more than two seats. In every run of my algorithm, 99.72% of comparison maps would allow Democrats to capture a third seat with a smaller wave.

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.987%	9	99.94%	17	99.9956%	25	99.83%
2	99.99%	10	99.907%	18	99.9957%	26	99.79%
3	99.929%	11	99.85%	19	99.8%	27	99.975%
4	99.88%	12	99.9912%	20	99.922%	28	99.85%
5	99.86%	13	99.77%	21	99.961%	29	99.83%
6	99.934%	14	99.89%	22	99.952%	30	99.92%
7	99.73%	15	99.87%	23	99.97%	31	99.946%
8	99.96%	16	99.72%	24	99.911%	32	99.961%



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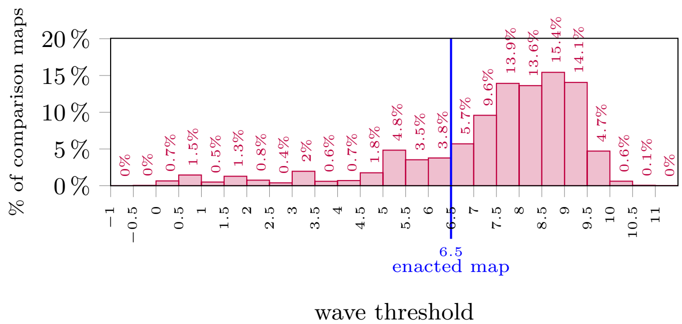
7.3 Cabarrus/Davie/Rowan/Yadkin

The comparison maps generated by my algorithm were similar to the enacted map with respect to their wave threshold for all seat values (results here shown for the wave threshold for 1 seat):

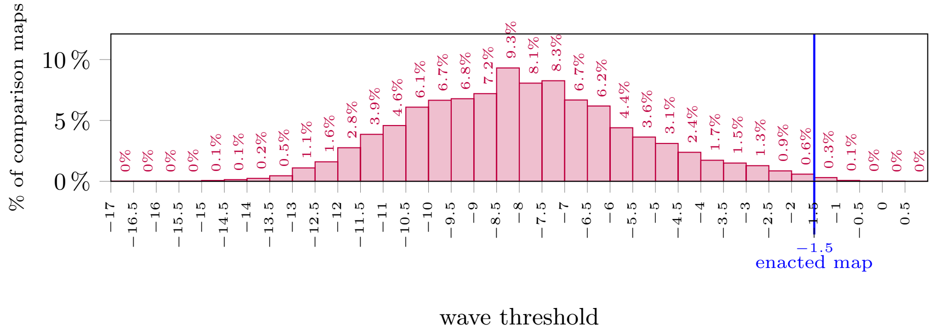
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	26.3%	9	20.7%	17	22.0%	25	22.3%
2	22.6%	10	23.1%	18	21.4%	26	20.8%
3	19.4%	11	27.6%	19	23.3%	27	20.2%
4	20.7%	12	21.2%	20	25.7%	28	22.0%
5	18.8%	13	23.4%	21	21.8%	29	22.1%
6	21.9%	14	25.4%	22	20.8%	30	22.3%
7	24.3%	15	20.0%	23	22.9%	31	22.4%
8	20.4%	16	19.9%	24	23.1%	32	23.8%



7.4 Cumberland

Despite the fact that my algorithm did not detect large differences between the enacted districting and comparison districtings of this cluster, the enacted map is an extreme outlier among the comparison maps generated by my algorithm with respect to the wave threshold for two seats.

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.61%	9	99.62%	17	99.62%	25	99.64%
2	99.64%	10	99.64%	18	99.62%	26	99.63%
3	99.61%	11	99.61%	19	99.61%	27	99.63%
4	99.62%	12	99.62%	20	99.63%	28	99.6%
5	99.59%	13	99.62%	21	99.64%	29	99.63%
6	99.61%	14	99.59%	22	99.63%	30	99.62%
7	99.61%	15	99.62%	23	99.62%	31	99.6%
8	99.61%	16	99.63%	24	99.62%	32	99.62%



Appendix A Multimoves / Precinct splits

As discussed in Section 5 my algorithm can be set to allow multiple changes to a map to occur in one step, when this is necessary to produce a sufficiently rich set of comparison maps.

Here I describe details of this technique so that technical experts can understand how precisely our method works. These details are not necessary to understand the basic mechanics of the method, which are simply that:

- Multiple changes may be made to a map in a single step,
- The result of the changes must always be a valid comparison map, in the sense that it complies with the districting criteria we consider in our report, and
- Our implementation of multiple moves does not bias the algorithm to any map or family of maps.

For technical experts: these multiple moves can be implemented with a Metropolis-Hastings approach. In particular, a score function based on the deviation of an invalid map from the compactness and population thresholds can be defined. *The score function is set to be equal for all maps satisfying the districting criteria.* With this choice, a uniform stationary distribution can be constructed on the space of maps satisfying the districting criteria. The Metropolis-Hastings chain will occasionally leave the feasible region of the map-space for some number of steps before returning to the feasible region. The collection of steps made outside the feasible region can be performed in a single step, to give a single multi-move which transforms one valid map into another valid map.

A related implementation detail concerns precinct splits. When operating at the geounit level but preserving the maximum number of precinct splits, I can allow the chain at intermediate points to have one more split than is allowed, while discarding these intermediate, invalid comparison maps. For example, in a map which currently splits two specific precincts, the chain is allowed to produce a valid comparison map by changing the district membership of another precinct. Note that this does not change the number of precinct splits, but viewed in terms of single geounit moves, it passes through a set of maps with a greater number of precinct splits. As in the case of multimoves discussed above, these intermediate maps are not part of the comparison set, and we can view the precinct swap as a single multimove of geounit swaps.

Finally, I note that when operating below the precinct level in House clusters with split precincts, my algorithm imposes an additional compactness-like constraint on any precinct splits, which is simply that the length of the precinct split is not large relative to the perimeter of the precinct itself. (The enacted plan satisfies this constraint in all cases.)

Appendix B Theorems

The second level analyses in my report are calculated using the theorems from [CFMP]; in particular, Theorem 1.5 from that manuscript suffices for all of my second-level findings here.

In plain language, that theorem says that if I conduct m runs, and observe that in every run the enacted plan is in the bottom ε fraction of comparison maps, then I can conclude that the enacted plan is among the most carefully crafted α fraction of *all* maps satisfying the districting criteria (not just those encountered by the algorithm), measured by their ε-fragility, at a statistical significance calculated with the formula

$$p = \left(\frac{2\varepsilon}{\alpha}\right)^{m/2}.$$

In this report, I frequently have $m = 32$ runs and choose α to simply be 3 times as big as ε. In this case, we see that we can conclude that the enacted plan is among the most carefully crafted 3ε of all maps, at a statistical significance of

$$p = \left(\frac{2}{3}\right)^{16} \approx .0015 < .002.$$

Note that, for example, if we used instead a threshold of $\alpha = 4\varepsilon$, this would give significance of

$$p = \left(\frac{2}{4}\right)^{16} \approx .000015,$$

and taking a threshold of $\alpha = 6\varepsilon$ would give

$$p = \left(\frac{2}{6}\right)^{16} \approx .00000002,$$

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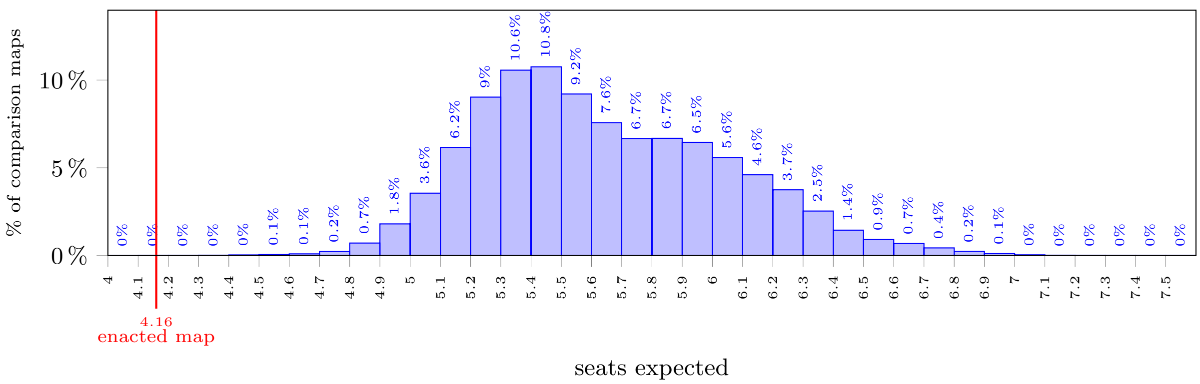
Appendix C Robustness Checks, Congressional districting

C.1 Robustness to election data

Here I show results when my analysis of the Congressional map is repeated with other elections in place of the 2020 Attorney General election as my proxy for partisan voting patterns.

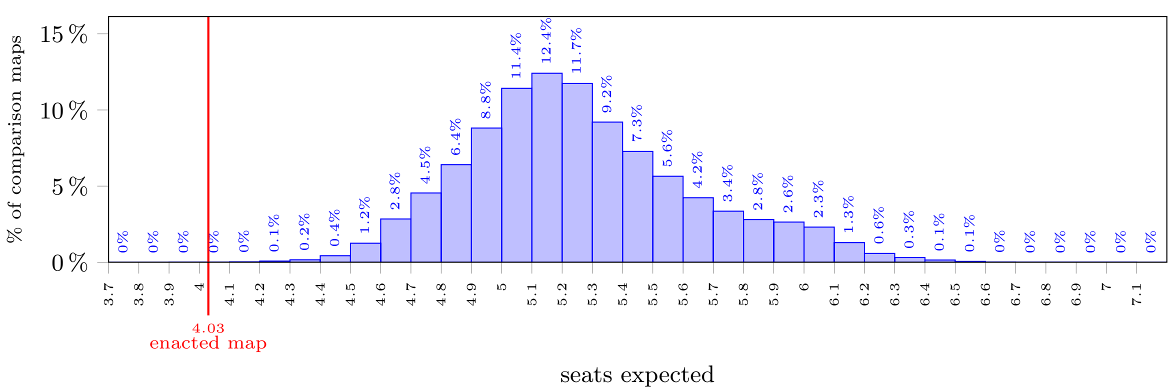
C.1.1 Results with 2020 Presidential election

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999925%	5	99.999986%	9	99.9999908%	13	99.9999926%
2	99.999921%	6	99.999999968%	10	99.9999932%	14	99.999988%
3	99.9999955%	7	99.999984%	11	99.9999979%	15	99.9999989%
4	99.9999933%	8	99.99995%	12	99.9999999981%	16	99.999978%



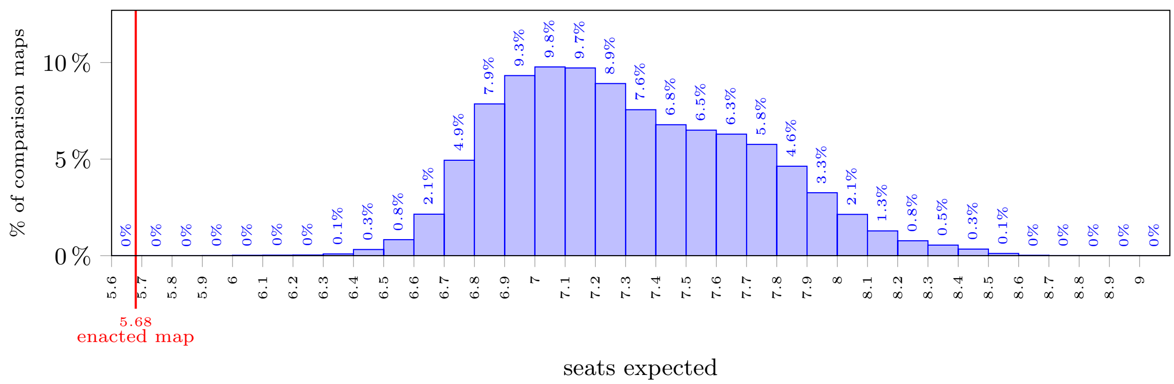
C.1.2 Results with 2020 Lieutenant Governor election

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.999973%	5	99.999937%	9	99.999942%	13	99.999982%
2	99.99985%	6	99.9999964%	10	99.99901%	14	99.999978%
3	99.999905%	7	99.99954%	11	99.9999928%	15	99.999934%
4	99.999964%	8	99.99975%	12	99.9995%	16	99.9998%



C.1.3 Results with 2020 Governor election

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999989%	5	99.9999979%	9	99.9999975%	13	99.99999923%
2	99.9999914%	6	99.9999999922%	10	99.99999974%	14	99.99999968%
3	99.9999996%	7	99.999999934%	11	99.999999994%	15	99.999999982%
4	99.999999966%	8	99.9999982%	12	99.9999999981%	16	99.999999961%

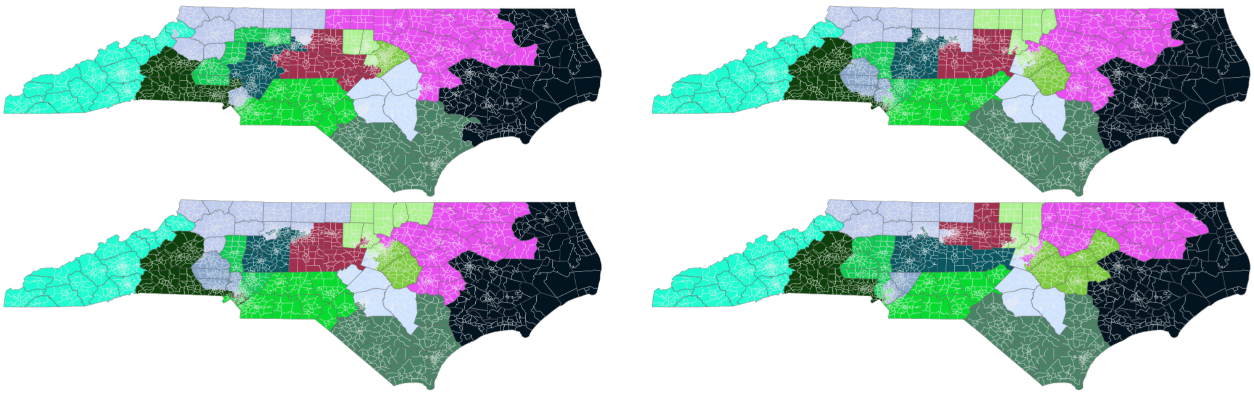


[Report continues on next page for formatting reasons]

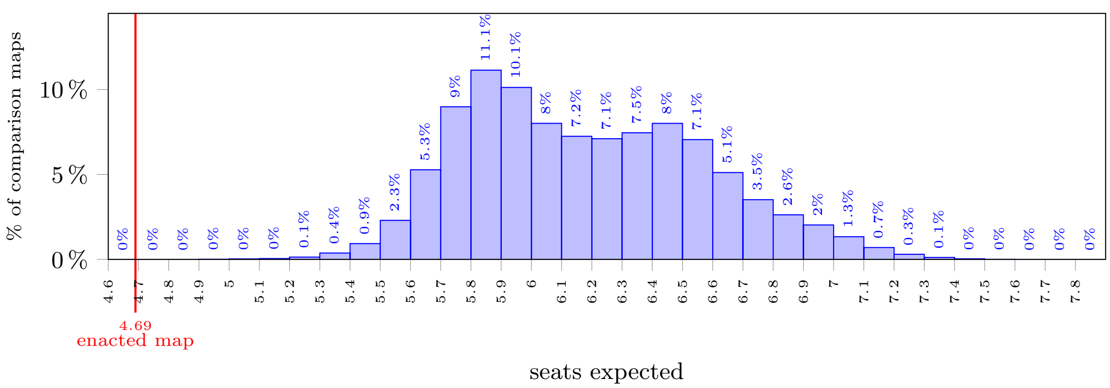
C.2 Robustness to incumbency protection

Here I show results when my analysis of the Congressional map is repeated without ensuring the protection of incumbents.

C.2.1 Comparison map examples



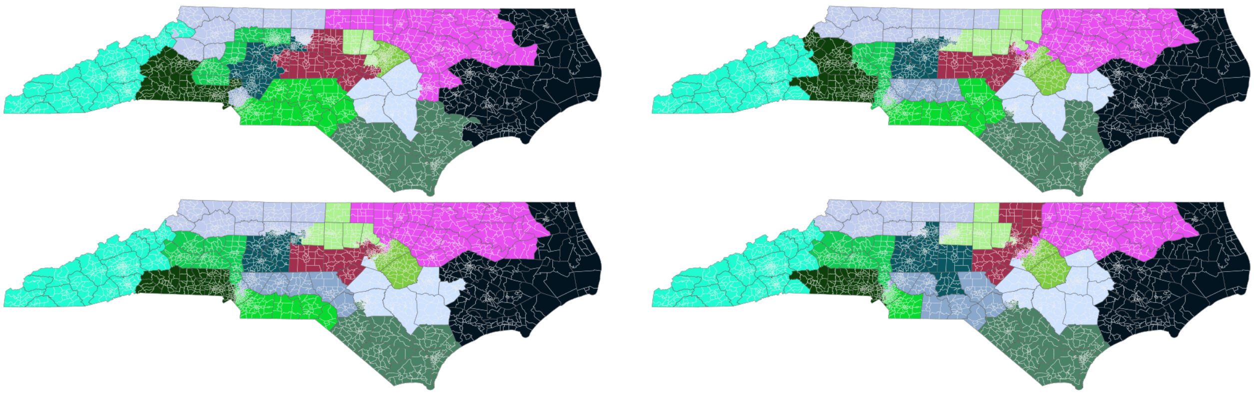
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.999998%	5	99.99999918%	9	99.9999976%	13	99.999982%
2	99.999999901%	6	99.9999978%	10	99.999989%	14	99.99999901%
3	99.9999986%	7	99.999999961%	11	99.9999967%	15	99.99999977%
4	99.9999967%	8	99.9999954%	12	99.9999999981%	16	99.9999986%



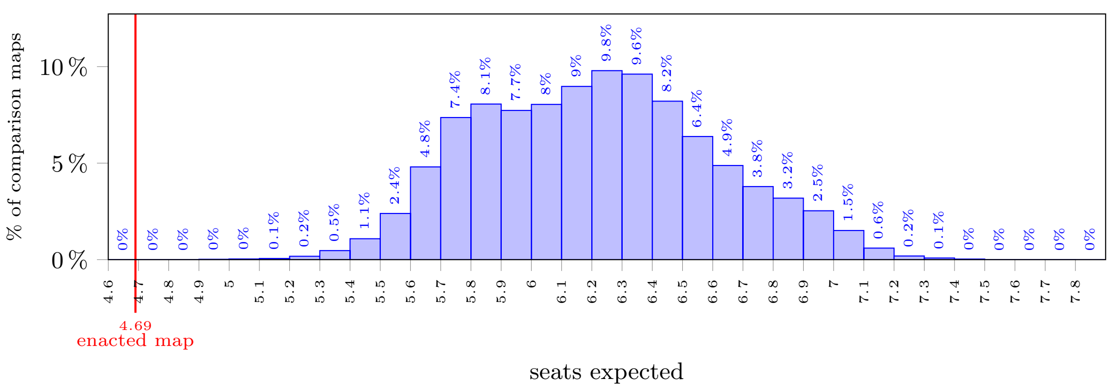
C.3 Robustness to compactness: 0% Polsby-Popper threshold

Here I show results when my analysis of the Congressional map is repeated with a 0% threshold for compactness in place of the 5% error I allow in my primary analysis.

C.3.1 Comparison map examples



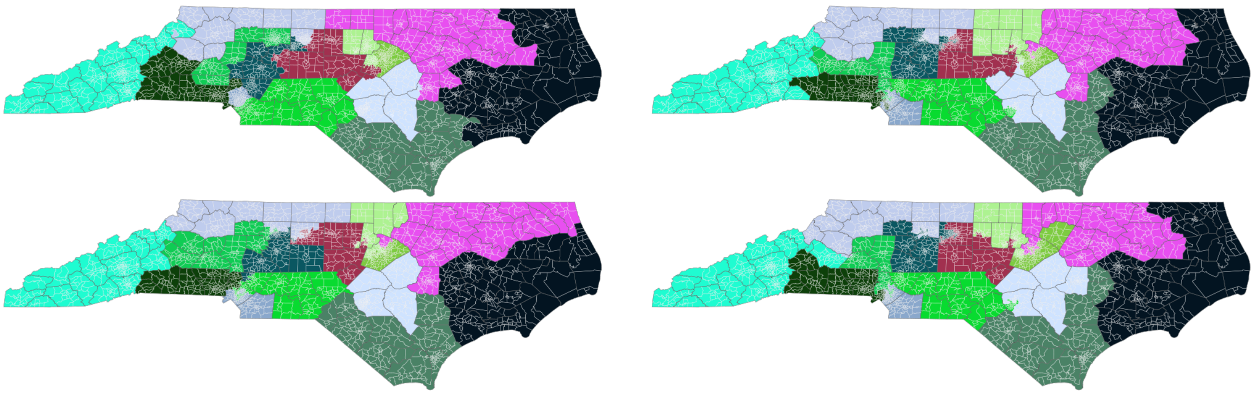
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.99999989%	5	99.9999997%	9	99.9999975%	13	99.999979%
2	99.9999984%	6	99.99999983%	10	99.9999968%	14	99.9999968%
3	99.9999933%	7	99.9999962%	11	99.9999968%	15	99.9999983%
4	99.999986%	8	99.9999983%	12	99.99999954%	16	99.9999984%



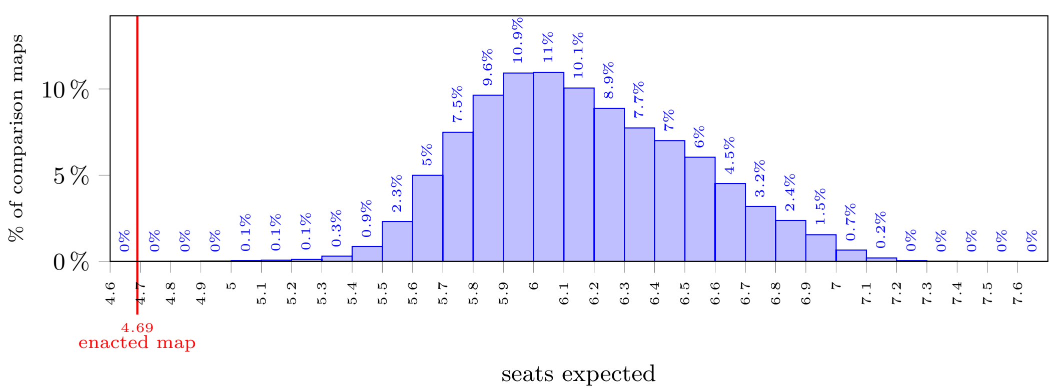
C.4 Robustness to compactness: 10% Polsby-Popper threshold

Here I show results when my analysis of the Congressional map is repeated with a 10% threshold for compactness in place of the 5% error I allow in my primary analysis.

C.4.1 Comparison map examples



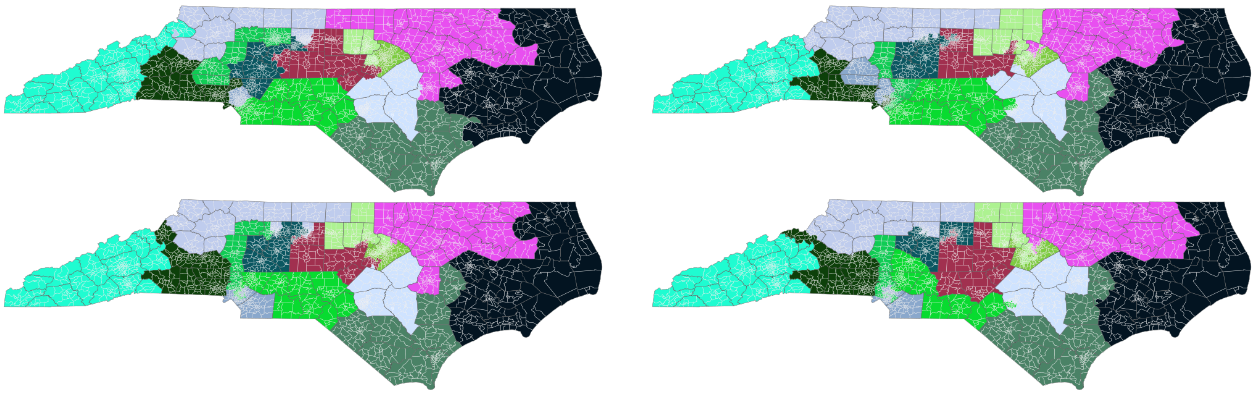
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999988%	5	99.9999974%	9	99.999982%	13	99.9999976%
2	99.9999989%	6	99.9999989%	10	99.9999954%	14	99.9999985%
3	99.9999961%	7	99.999999946%	11	99.9999965%	15	99.99999983%
4	99.99999981%	8	99.9999973%	12	99.9999999981%	16	99.99999985%



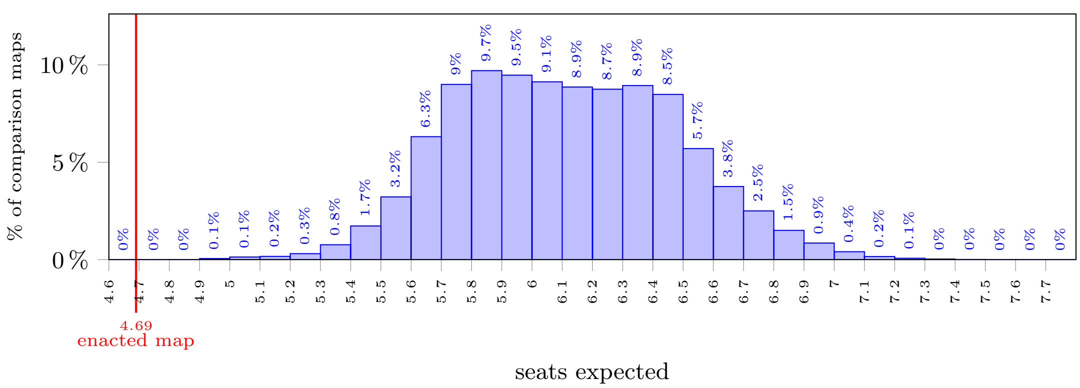
C.5 Robustness to compactness 5% Perimeter compactness

Here I show results when my analysis of the Congressional map is repeated with a completely different compactness score, based just on the total perimeter of all districts in the districting.

C.5.1 Comparison map examples



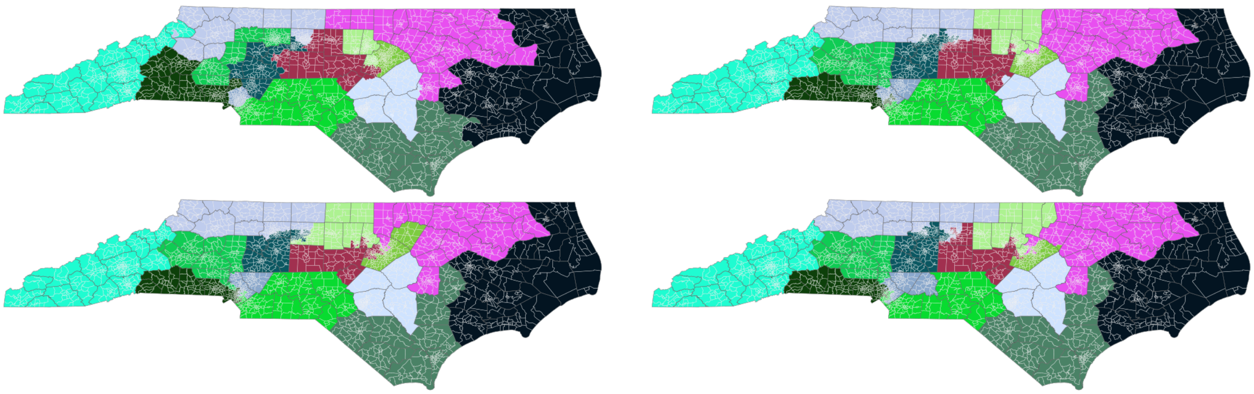
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999988%	5	99.9999968%	9	99.999998%	13	99.9999976%
2	99.99999948%	6	99.9999949%	10	99.9999978%	14	99.9999986%
3	99.99999941%	7	99.9999999976%	11	99.999982%	15	99.99999983%
4	99.99999981%	8	99.9999906%	12	99.9999999981%	16	99.9999963%



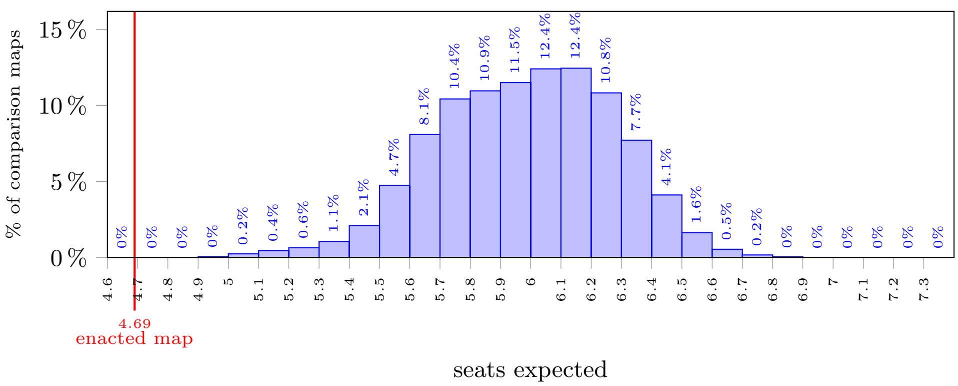
C.6 Robustness to 1% population deviation

Here I show results when my analysis of the Congressional map is repeated with a 1% population deviation constraint instead of a 2% population deviation constraint.

C.6.1 Comparison map examples



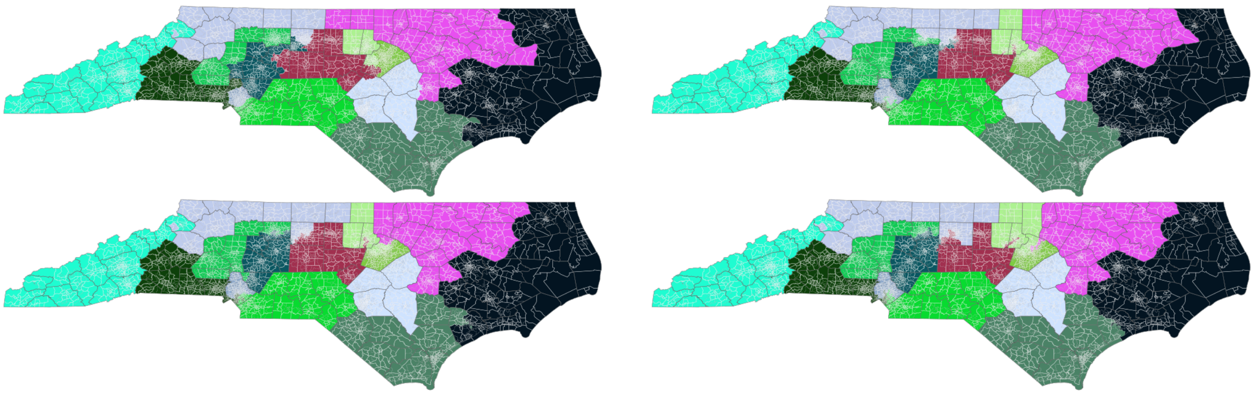
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999911%	5	99.999999907%	9	99.9999983%	13	99.999914%
2	99.9999966%	6	99.99999999945%	10	99.99978%	14	99.9999988%
3	99.999949%	7	99.9999986%	11	99.999989%	15	99.999971%
4	99.9999935%	8	99.999951%	12	99.999934%	16	99.999997%



C.7 Geounit analysis

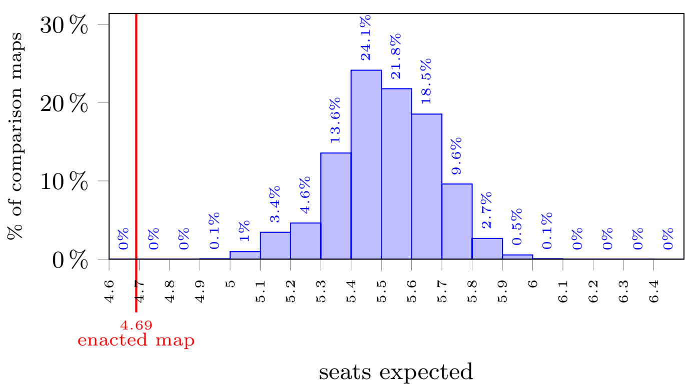
Here I show results when my analysis of the Congressional map is repeated at the geounit level, with a 0.5% population deviation constraint.

C.7.1 Comparison map examples



C.7.2 Results

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.999952%	5	99.999987%	9	99.999962%	13	99.9999952%
2	99.999989%	6	99.999986%	10	99.9999964%	14	99.9999962%
3	99.999967%	7	99.9999924%	11	99.999974%	15	99.999926%
4	99.999964%	8	99.999996%	12	99.999977%	16	99.9999935%

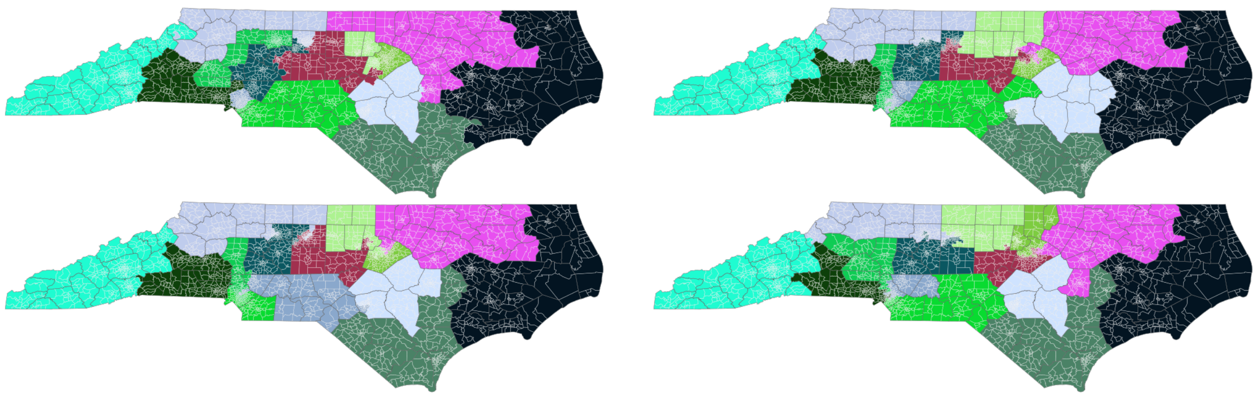


- **First level analysis:** In *every* run, the districting was in the most partisan 0.000073% of districtings (in other words, 99.999926% were less partisan, in *every* run).
- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.00022% of all alternative districtings of North Carolina satisfying my districting criteria (in other words, 99.99977% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.000073\%$.

C.8 Analysis of VTD-level blueprint

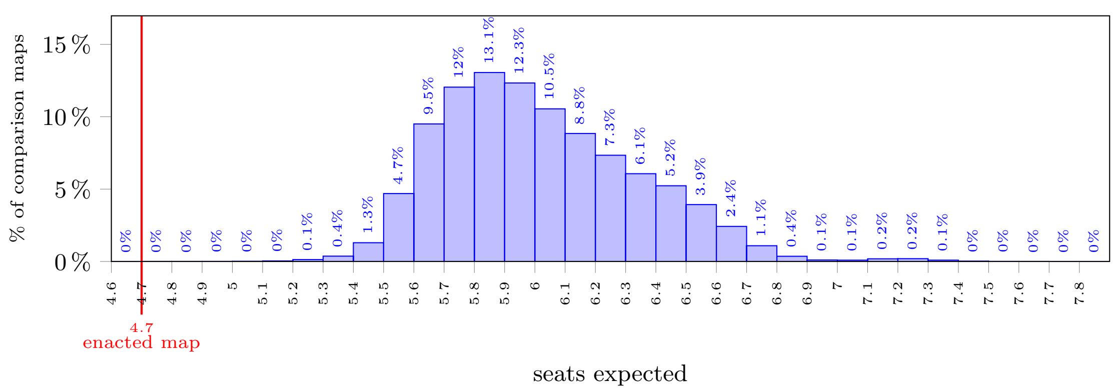
Here I show results when my analysis of the Congressional map is performed not on the precise enacted map, but a whole-VTD-level blueprint for the enacted map obtained by assigning each split VTD to the district it has the greatest intersection with.

C.8.1 Comparison map examples



C.8.2 Results

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999982%	9	99.99999969%	17	99.9999991%	25	99.9999986%
2	99.99999947%	10	99.9999952%	18	99.99999944%	26	99.9999998%
3	99.9999957%	11	99.999986%	19	99.999978%	27	99.9999977%
4	99.9999907%	12	99.999979%	20	99.9999959%	28	99.9999976%
5	99.9999981%	13	99.9999986%	21	99.99999946%	29	99.99999958%
6	99.99999954%	14	99.999984%	22	99.9999971%	30	99.999986%
7	99.9999917%	15	99.9999977%	23	99.9999974%	31	99.9999969%
8	99.9999917%	16	99.9999961%	24	99.9999942%	32	99.9999958%



- **First level analysis:** In *every* run, the districting was in the most partisan 0.000021% of districtings (in other words, 99.999978% were less partisan, in *every* run).
- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.000064% of all alternative districtings of North Carolina satisfying my districting criteria (in other words, 99.999935% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.000021\%$.

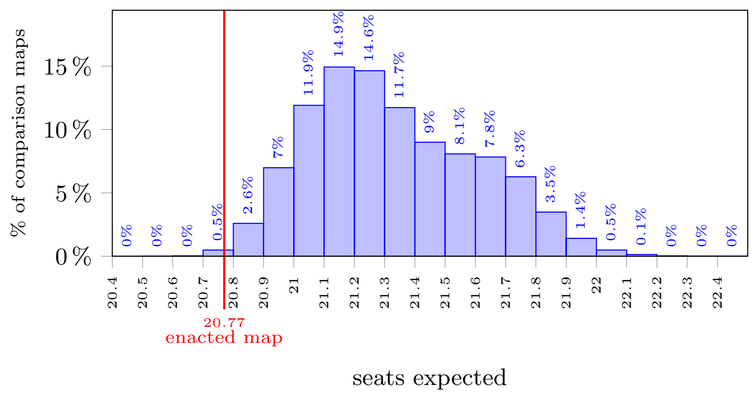
Appendix D Robustness Checks, Senate districting

D.1 Robustness to election data

Here I show results when my analysis of the Senate map is repeated with other elections in place of the 2020 Attorney General election as my proxy for partisan voting patterns.

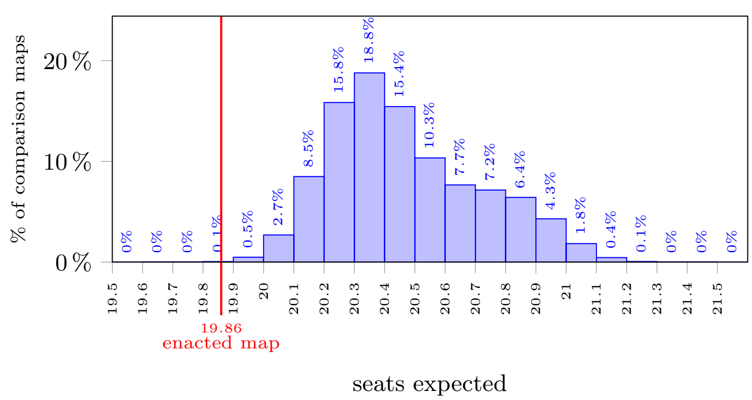
D.1.1 Results with 2020 Presidential election

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.65%	5	99.78%	9	99.79%	13	99.8%
2	99.81%	6	99.79%	10	99.82%	14	99.73%
3	99.75%	7	99.79%	11	99.81%	15	99.66%
4	99.8%	8	99.75%	12	99.75%	16	99.81%



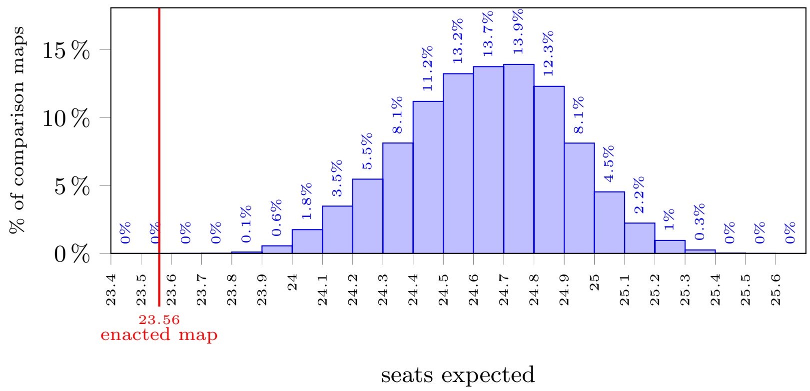
D.1.2 Results with 2020 Lieutenant Governor election

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.943%	5	99.987%	9	99.9912%	13	99.9911%
2	99.996%	6	99.982%	10	99.9955%	14	99.977%
3	99.973%	7	99.994%	11	99.9958%	15	99.944%
4	99.9927%	8	99.983%	12	99.89%	16	99.995%



D.1.3 Results with 2020 Governor election

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.99999936%	5	99.9999996%	9	99.9999998%	13	99.999999973%
2	99.999999949%	6	99.9999974%	10	99.9999987%	14	99.9999985%
3	99.99999978%	7	99.9999999929%	11	99.9999998%	15	99.999999961%
4	99.9999989%	8	99.9999999969%	12	99.999999973%	16	99.9999985%



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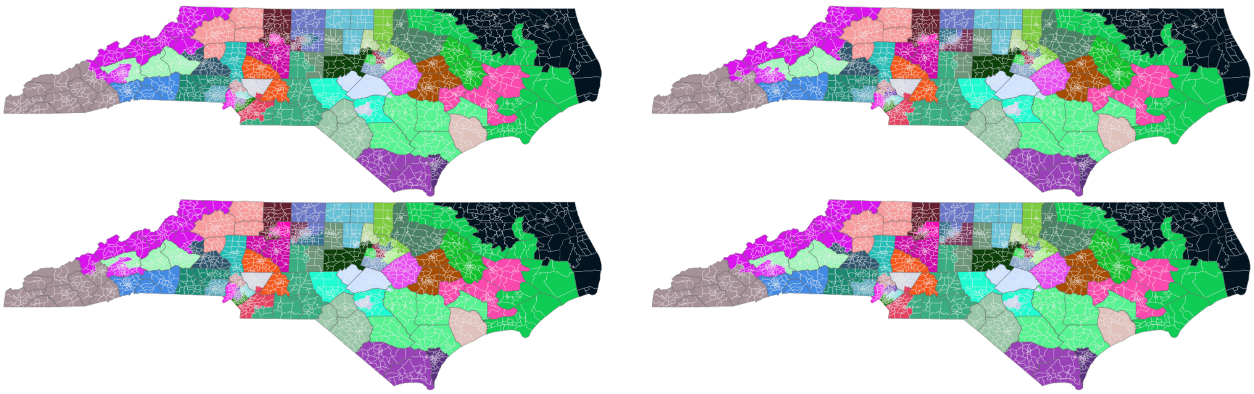
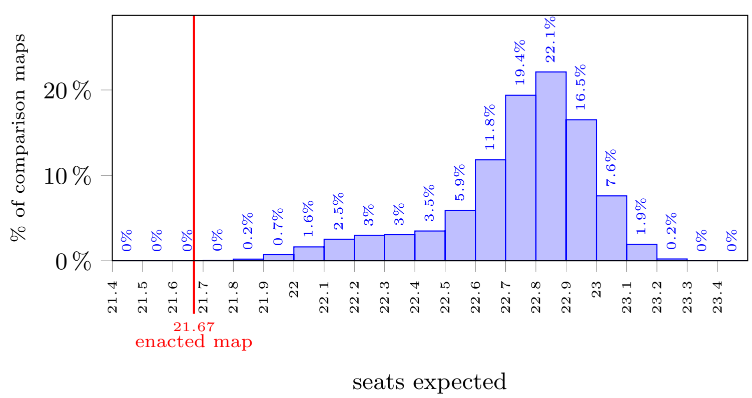
D.2 Robustness to incumbency protection

Here I show results when my analysis of the Senate map is repeated without ensuring the protection of incumbents.

D.2.1 Comparison map examples



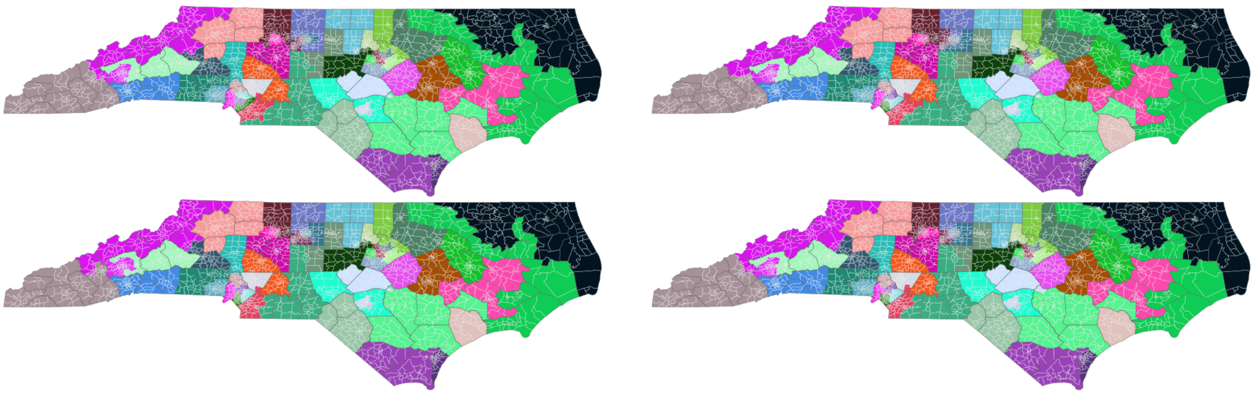
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9998%	5	99.9993%	9	99.99989%	13	99.99906%
2	99.99988%	6	99.99985%	10	99.99968%	14	99.9987%
3	99.99971%	7	99.999907%	11	99.9998%	15	99.99928%
4	99.99922%	8	99.9985%	12	99.99976%	16	99.9943%



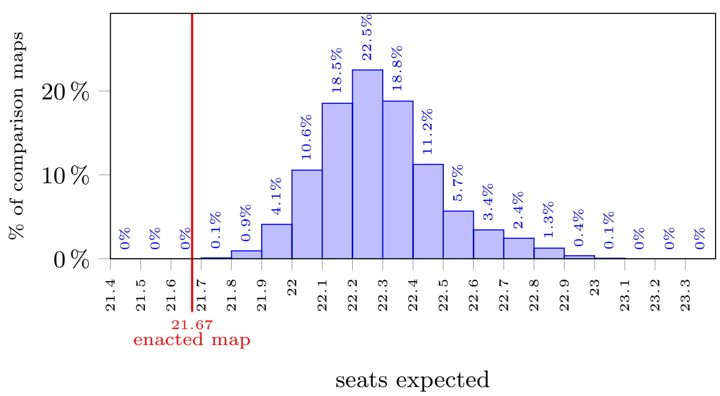
D.3 Compactness: 0% Polsby-Popper threshold

Here I show results when my analysis of the Senate map is repeated with a 0% threshold for compactness in place of the 5% error I allow in my primary analysis.

D.3.1 Comparison map examples



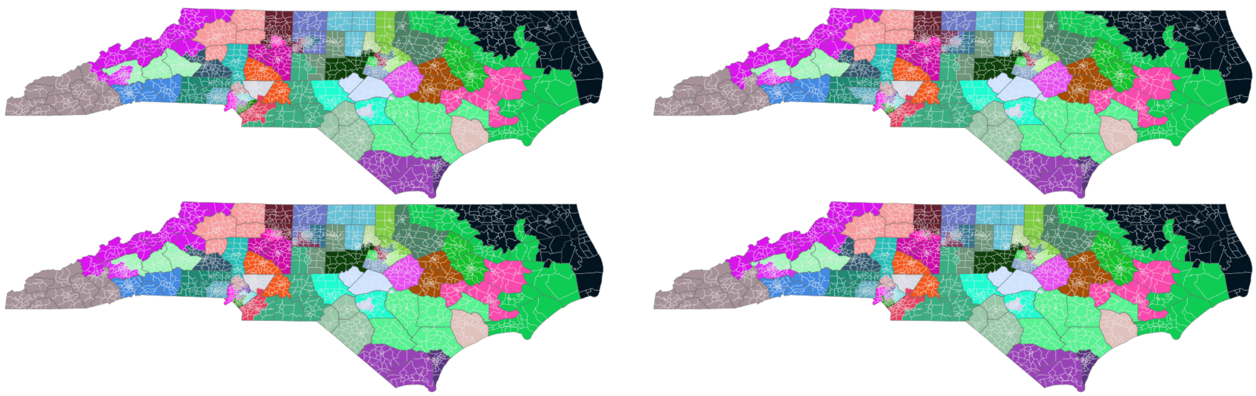
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9979%	5	99.9978%	9	99.995%	13	99.9986%
2	99.99909%	6	99.9968%	10	99.9982%	14	99.9989%
3	99.9968%	7	99.99933%	11	99.9987%	15	99.9973%
4	99.99927%	8	99.9979%	12	99.99923%	16	99.9976%



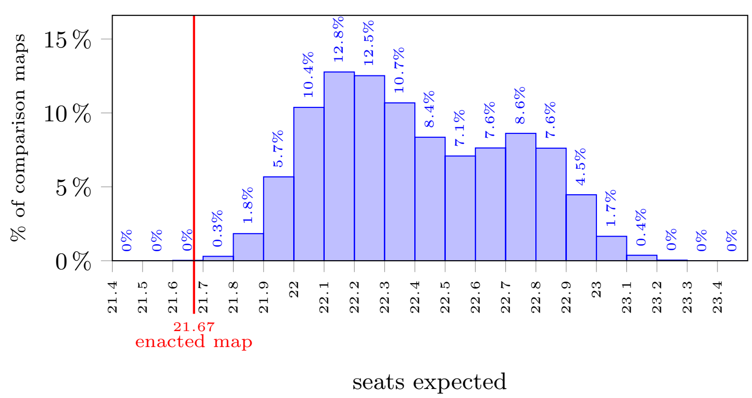
D.4 Compactness: 10% Polsby-Popper threshold

Here I show results when my analysis of the Senate map is repeated with a 10% threshold for compactness in place of the 5% error I allow in my primary analysis.

D.4.1 Comparison map examples



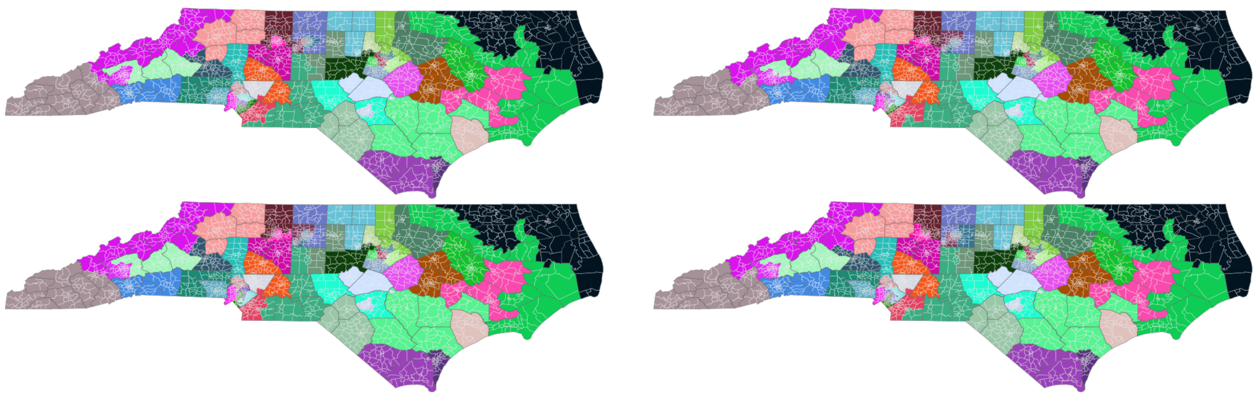
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9963%	5	99.992%	9	99.971%	13	99.98%
2	99.9928%	6	99.986%	10	99.985%	14	99.9917%
3	99.988%	7	99.993%	11	99.9924%	15	99.978%
4	99.987%	8	99.9957%	12	99.9908%	16	99.9969%



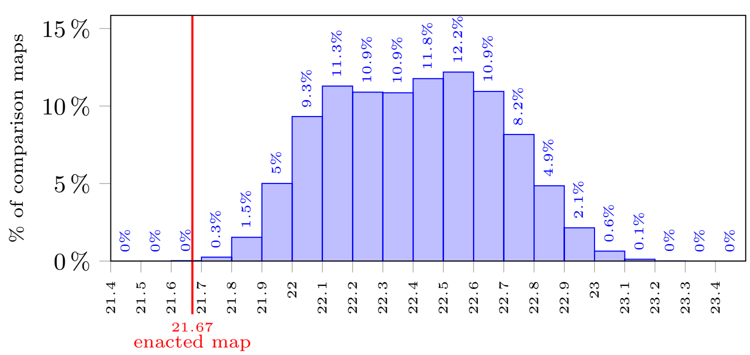
D.5 Compactness 5% Perimeter compactness

Here I show results when my analysis of the Senate map is repeated with a completely different compactness score, based just on the total perimeter of all districts in the districting.

D.5.1 Comparison map examples



Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9913%	5	99.985%	9	99.988%	13	99.9907%
2	99.9907%	6	99.989%	10	99.988%	14	99.982%
3	99.9949%	7	99.9929%	11	99.986%	15	99.981%
4	99.989%	8	99.989%	12	99.987%	16	99.9919%



seats expected

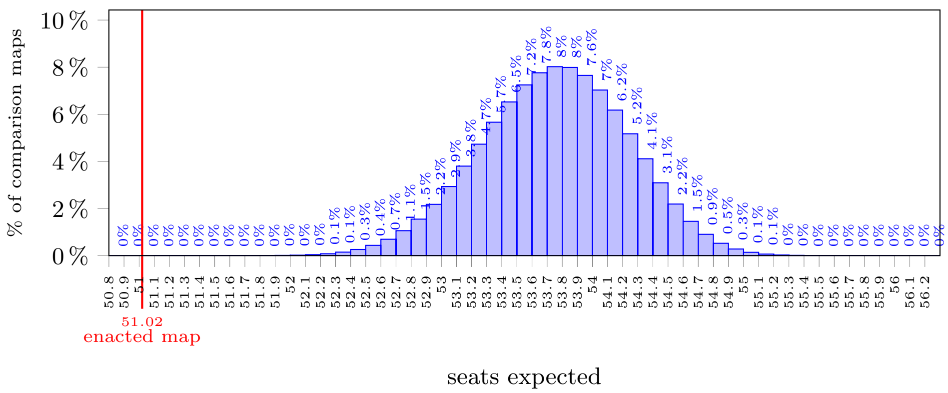
Appendix E Robustness Checks, House districting

E.1 Robustness to election data

Here I show results when my analysis of the House map is repeated with other elections in place of the 2020 Attorney General election as my proxy for partisan voting patterns.

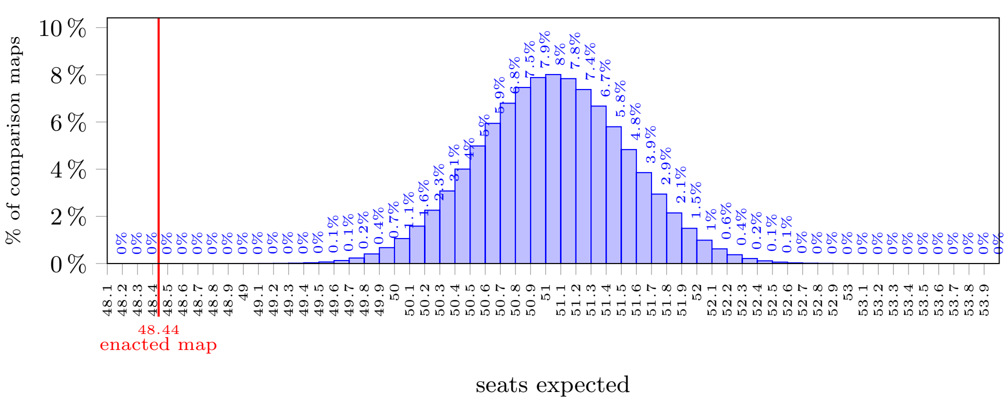
E.1.1 Results with 2020 Presidential election

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.999999985%	5	99.99999945%	9	99.9999986%	13	99.99999986%
2	99.999999981%	6	99.99999948%	10	99.99999912%	14	99.999999976%
3	99.99999997%	7	99.999999963%	11	99.99999986%	15	99.99999984%
4	99.9999969%	8	99.9999981%	12	99.9999985%	16	99.9999989%



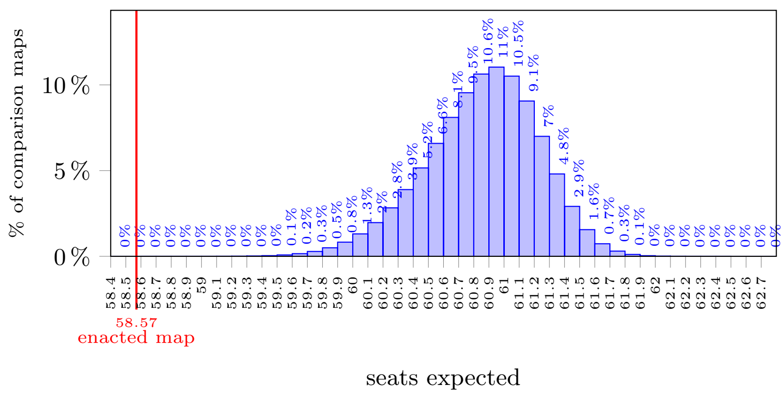
E.1.2 Results with 2020 Lieutenant Governor election

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.99999988%	5	99.9999983%	9	99.999997%	13	99.9999957%
2	99.999981%	6	99.9999926%	10	99.9999979%	14	99.9999905%
3	99.99999907%	7	99.9999927%	11	99.9999974%	15	99.99999914%
4	99.9999969%	8	99.999993%	12	99.9999981%	16	99.99999924%



E.1.3 Results with 2020 Governor election

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999985%	5	99.999999931%	9	99.999999975%	13	99.99999986%
2	99.999999984%	6	99.9999994%	10	99.9999986%	14	99.99999988%
3	99.99999997%	7	99.999999986%	11	99.9999998%	15	99.99999948%
4	99.9999985%	8	99.99999985%	12	99.99999914%	16	99.99999989%

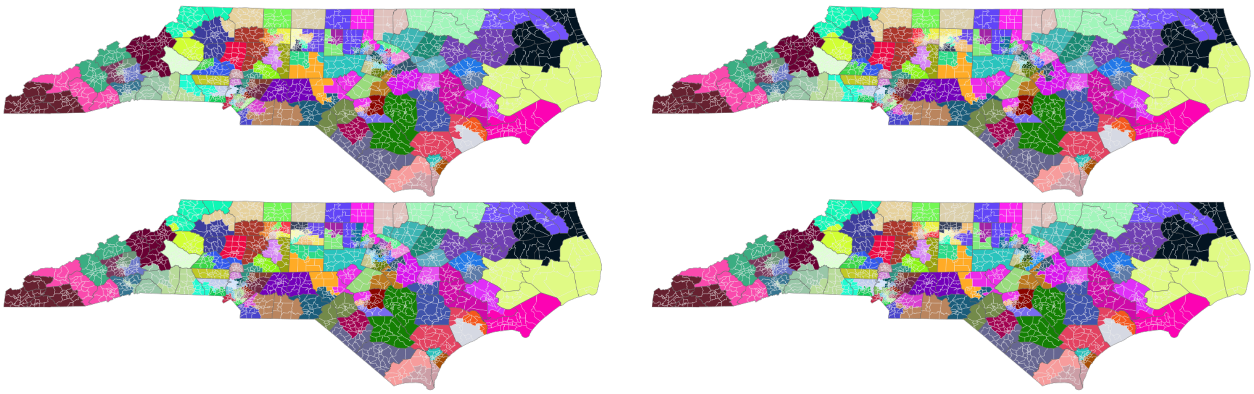


[Report continues on next page for formatting reasons]

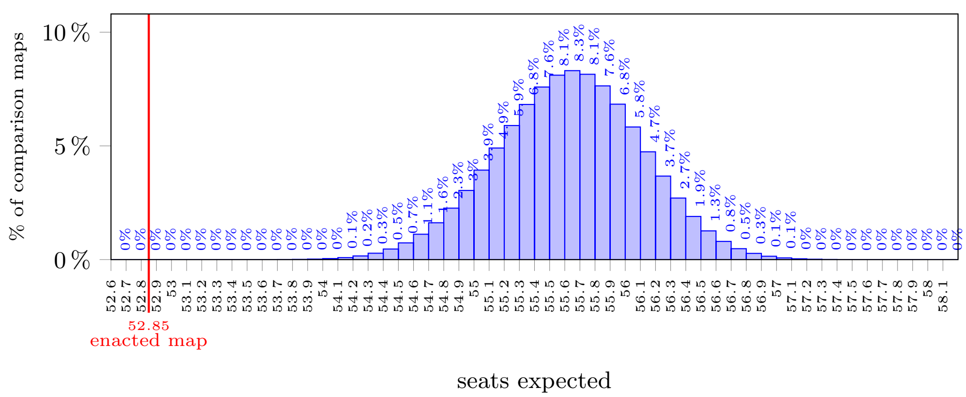
E.2 Robustness to incumbency protection

Here I show results when my analysis of the House map is repeated without ensuring the protection of incumbents.

E.2.1 Comparison map examples



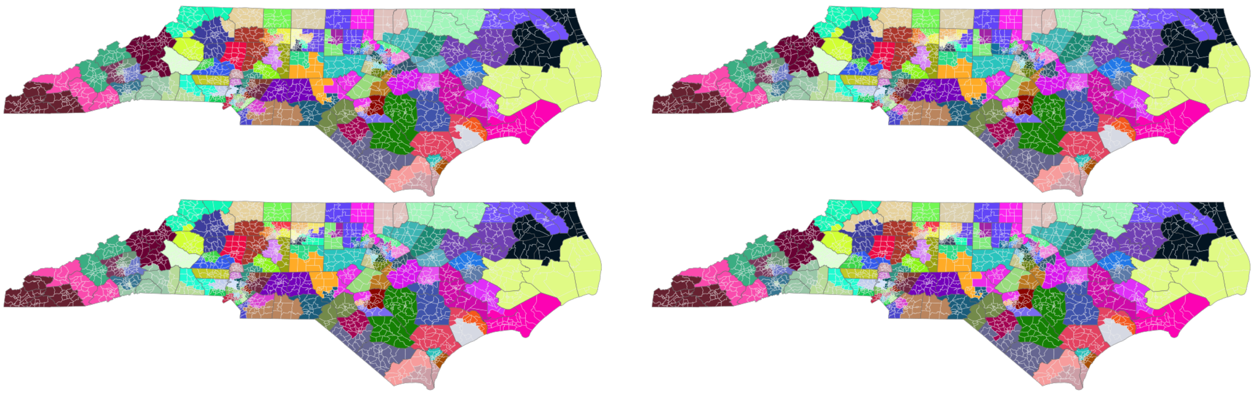
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.99999987%	5	99.9999933%	9	99.99999967%	13	99.99999989%
2	99.999999981%	6	99.9999962%	10	99.99999944%	14	99.99999981%
3	99.99999997%	7	99.9999968%	11	99.9999944%	15	99.99999%
4	99.999999908%	8	99.99999961%	12	99.999999963%	16	99.99999947%



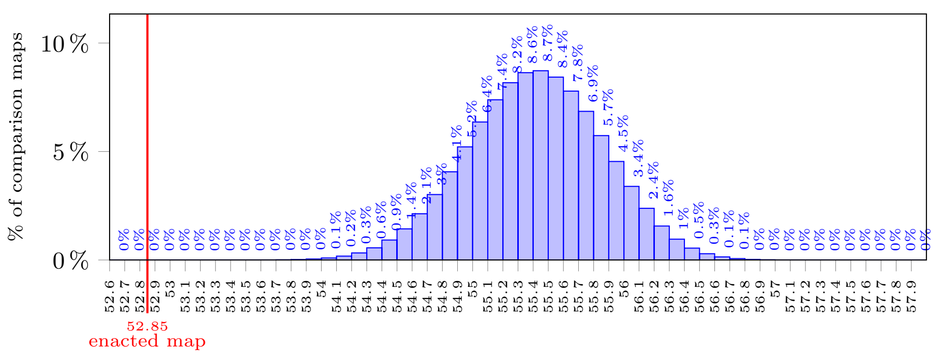
E.3 Compactness: 0% Polsby-Popper threshold

Here I show results when my analysis of the House map is repeated with a 0% threshold for compactness in place of the 5% error I allow in my primary analysis.

E.3.1 Comparison map examples



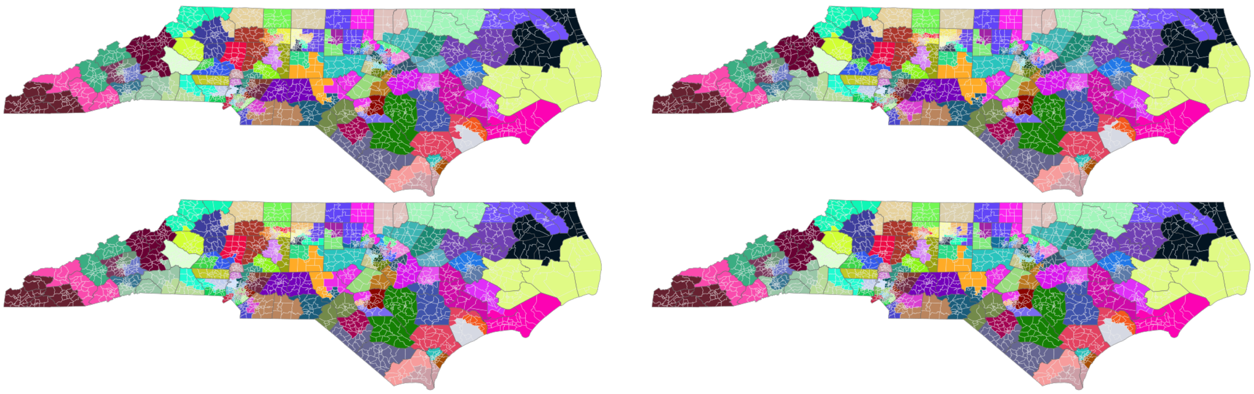
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999996%	5	99.99999927%	9	99.9999987%	13	99.9999978%
2	99.99999982%	6	99.999999941%	10	99.9999966%	14	99.9999986%
3	99.999987%	7	99.9999971%	11	99.9999963%	15	99.99999975%
4	99.9999912%	8	99.9999988%	12	99.99999928%	16	99.9999968%



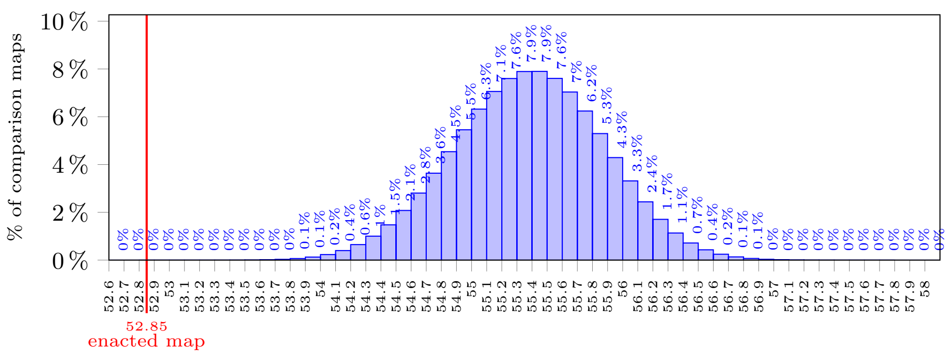
E.4 Compactness: 10% Polsby-Popper threshold

Here I show results when my analysis of the House map is repeated with a 10% threshold for compactness in place of the 5% error I allow in my primary analysis.

E.4.1 Comparison map examples



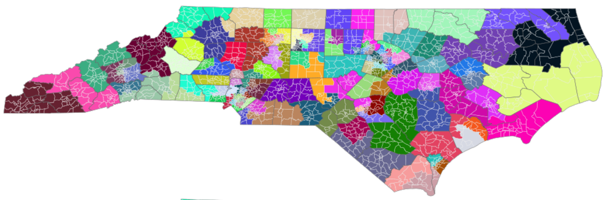
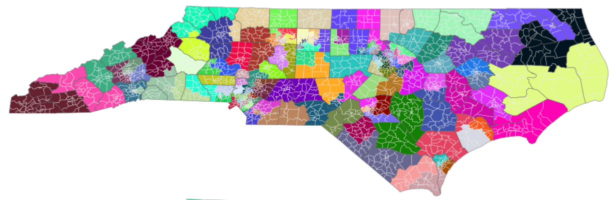
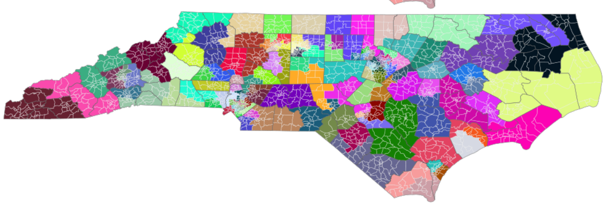
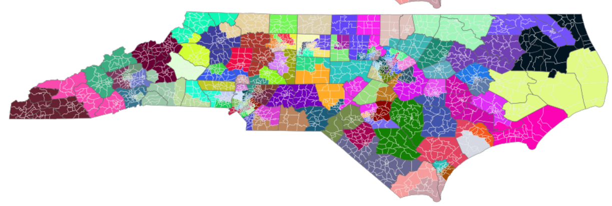
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999904%	5	99.9999989%	9	99.999999917%	13	99.9999983%
2	99.999999957%	6	99.9999971%	10	99.9999983%	14	99.99999989%
3	99.9999948%	7	99.9999999916%	11	99.999988%	15	99.99999962%
4	99.9999987%	8	99.9999955%	12	99.9999922%	16	99.9999974%



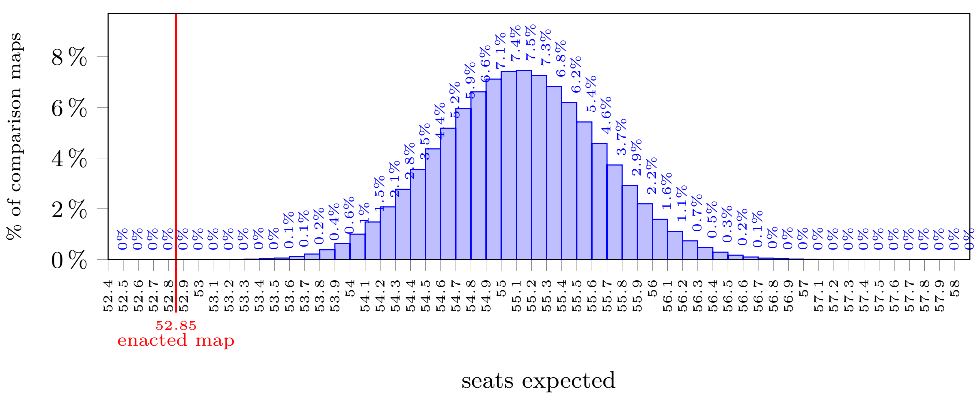
E.5 Compactness 5% Perimeter compactness

Here I show results when my analysis of the House map is repeated with a completely different compactness score, based just on the total perimeter of all districts in the districting.

E.5.1 Comparison map examples

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.99985%	5	99.999957%	9	99.999988%	13	99.999953%
2	99.999977%	6	99.999976%	10	99.999978%	14	99.99991%
3	99.99988%	7	99.9999904%	11	99.999968%	15	99.999981%
4	99.999978%	8	99.999951%	12	99.999925%	16	99.99995%



I hereby certify that the foregoing statements are true and correct to the best of my knowledge, information, and belief.

Wesley Pegden
12/23/2021